Differential Geometry III: Riemannian Geometry

A lecture by Prof. Giorgios Moschidis at EPFL

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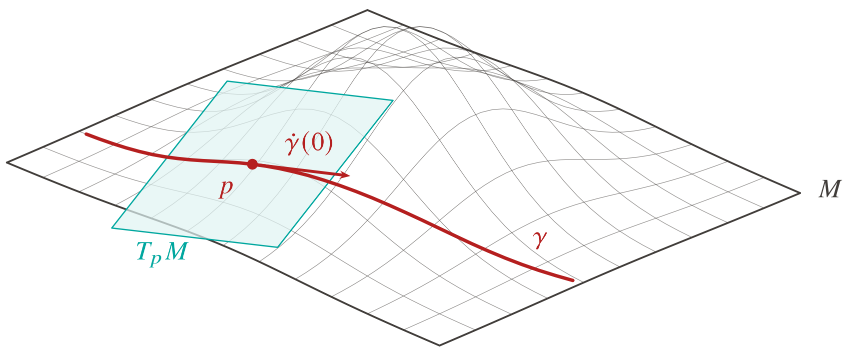

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1 Introduction

These are the lecture notes based on the lecture MATH-344 about Riemannian geometry given by Prof. Giorgios Moschidis at EPFL during the spring semester of 2026. The lecture covers the following topics:

- Riemannian metrics: Inner products on tangent spaces enabling length and distance measurements
- Connections and geodesics: Generalizing straight lines to curved manifolds
- Curvature tensors: Quantifying how manifolds bend
- Metric structure: Including the Hopf-Rinow theorem relating geometry to topology
- Hypersurfaces: Geometry of submanifolds
- Comparison theorems: Relating geometry to topology

The course extends Euclidean geometry concepts to the broader setting of Riemannian manifolds, essential for applications in PDEs, differential geometry, and theoretical physics. As a reference, the reader may consult [Car92]. The prerequisite is differential topology at the level of the Differential Geometry II course at EPFL, [TR24], which is based on [Lee03]. Another popular reference is [Lee18].

2 Short review of Differential Topology

(Lecture 1)

The following section is a quick revision of the material covered in Differential Geometry II at EPFL, [TR24].

> **Definition 2.1** (Smooth Manifold). A differentiable (or smooth) manifold of dimension n is a Hausdorff, second countable topological space M such that there exists a collection of open sets $\{U_a\}_{a\in A}$ of M and homeomorphisms $\phi_a : U_a \to \phi_a(U_a) \subseteq \mathbb{R}^n$ with the following properties:
>
> 1. $\bigcup_{a\in A} U_a = M$
> 2. For all $a, b \in A$ such that $U_a \cap U_b \neq \emptyset$, the map $\phi_a \circ \phi_b^{-1}$ is a smooth diffeomorphism between open subsets of $\mathbb{R}^n$.
> 3. The collection $\{U_a, \phi_a\}$ is maximal.
>
> We will write M^n to indicate the dimension of the manifold.

Remark 2.2. We recall some facts.

- The dimension is locally constant (invariance of domain theorem).
- We call $\{U_a, \phi_a\}$ a coordinate chart. and ϕ_a^{-1} a parametrisation.
- We call $(x^1, \ldots, x^n) = \phi_a$ or more explicitely $x^i : U_a \to \mathbb{R}, p \mapsto [\phi_a(p)]^i$ the local coordinates.
- $\phi_1 \circ \phi_2^{-1}$ is called a transition map or coordinate transformation.
- $\{U_a, \phi_a\}_{a\in A}$ is called an atlas, if it satisfies 1) and 2). If it also satisfies 3), we call it a differentiable structure.
- The Hausdorff property is needed to guarantee the uniqueness of limits, which is important for defining continuity and smoothness.

- The second countability condition (existence of countable basis of open sets) is needed to guarantee the existence of partitions of unity,

Definition 2.3 (Partition of Unity). Let $\{U_a, \phi_a\}$ be an atlas on a mfd. M. A partition of unity subordinate to $\{U_a, \phi_a\}$ is a collection of smooth functions $\{\chi_\beta : M \to \mathbb{R}\}_{\beta \in B}$ such that:

1. $\forall \beta \exists \alpha$ s.t. $\operatorname{supp} \chi_\beta \subset U_\alpha$
2. For all β, $\operatorname{supp} \chi_\beta \cap \operatorname{supp} \chi_\gamma \neq \emptyset$ for finitely many γ.
3. $\sum_\beta \chi_\beta = 1$, which is a finite sum, due to 2.

Remark 2.4. In our settings, partitions of unity always exist. We only consider manifolds that are locally Euclidean, Hausdorff, and second countable. This implies paracompactness, which in turn implies the existence of partitions of unity.

Definition 2.5 (Smooth function). A function $f : M^m \to N^n$ is called smooth if its expression in any local coordinate charts $\tilde{f} = \psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is smooth. The same applies to continuity, Hölder continuity, and continuous differentiability of order k.

Remark 2.6. It is enough to check the smoothness of f in one local coordinate chart, due to the smoothness of transition maps.
If the transition maps were merely C^k, we could only talk about C^k functions at most.

Functions (and in fact also all covariant tensors, like k-forms) can be pulled back via smooth maps while vector fields (and in fact also all contravariant tensors, like vector fields) can be pushed forward via certain smooth maps.

Definition 2.7 (Pullback of a function). If $f : M \to N$ is smooth, then for all $h : N \to \mathbb{R}$, we can always define the pullback of h via f by

$$\begin{aligned} f^*h : M &\to \mathbb{R} \\ p &\mapsto h \circ f(p). \end{aligned}$$

The pullback $f^* : C^k(N) \to C^k(M)$ is an $\mathbb{R}$-linear map.
We will introduce tangent vectors now. We can view a tangent vector as a direction of differentiation or as a differential operator of a smooth function.

Definition 2.8 (Tangent and Cotangent Space). Let M be a smooth manifold of dim n, $p \in M$, and $f, g \in C^\infty(M)$. A tangent vector v at p is a linear map $v : C^\infty(M) \to \mathbb{R}$ satisfying the Leibniz rule

$$v(fg) = v(f)g(p) + f(p)v(g)$$

The $\mathbb{R}$-vector space of tangent vectors at p is then denoted by T_pM.

The space T_pM has dimension n, with a natural basis given by the tangent vectors along the coordinate curves.

Definition 2.9 (Coordinate basis vectors). Let M, n, p, f, g, v be as above. We define the coordinate basis vector

$$\left.\frac{\partial}{\partial x^i}\right|_p f := \left.\left(\frac{\partial}{\partial x^i}(f \circ \varphi)\right)\right|_{\varphi(p)}.$$

Lemma 2.10 (Basis of T_pM). The set $\left\{ \frac{\partial}{\partial x^i}\Big|_p \right\}_{i=1}^n$ is a basis of T_pM.

Proof. See chapter 3 in [TR24] □

We define the dual notation.

Definition 2.11 (Cotangent space and differential of a function). The dual space of T_pM is called the cotangent space and is denoted by T_p^*M. For $f \in C^\infty(M)$, the differential of f at p is the linear map $\mathrm{d}f_p : T_pM \to \mathbb{R}$ defined by

$$\mathrm{d}f_p(v) := v(f).$$

So we have $\mathrm{d}f_p \in T_p^*M$.

Lemma 2.12 (Basis of T_p^*M). The set $\{\mathrm{d}x^i|_p\}_{i=1}^n$ consisting of the differentials of the coordinate functions $x^i : U \subset M \to \mathbb{R}$ is a basis of T_p^*M.

Proof. See chapter 8 and appendix C in [TR24] □

In local coordinates: $\{\mathrm{d}x^i\}_{i=1}^n$ is the dual basis of $\left\{\frac{\partial}{\partial x^i}\right\}_{i=1}^n$, as we have

$$\mathrm{d}x^i\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial}{\partial x^j}(x^i) = \delta^i_j.$$

In the following we will use the Einstein summation convection, which implies summation over repeated indices. For example, we write

$$v = v^i \frac{\partial}{\partial x^i}$$

instead of

$$v = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}$$

as the dimension n is usually clear from the context.

Definition 2.13 (Vector Field and Covector Field / 1-form). A vector field is an assignment/map $X : p \mapsto v \in T_pM$ that is smooth in local coordinates/when written in the coordinate basis

$$v = v^i \frac{\partial}{\partial x^i}.$$

We write $\Gamma(TM)$ for the set of all vector fields on M.

A covector-field or (differential) 1-form is an assignment/map $\omega : p \mapsto \omega_p \in T_p^*M$ that is smooth in local coordinates/when written in the coordinate basis

$$\omega_p = \omega_i \mathrm{d}x^i|_p.$$

We write $\Gamma(T^*M)$ for the set of all covector fields on M.

The next result is easy to show, but nevertheless important.

Lemma 2.14 (Change of Coordinates for vectors and covectors).

$$dy^i = \frac{\partial y^i}{\partial x^j} dx^j$$

$$\frac{\partial}{\partial x^i} = \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j}$$

Proof. See chapter 3 in [TR24]. □

Remark 2.15. This allows to translate a tangent vector (field)

$$v = v^i \frac{\partial}{\partial x^i} = \tilde{v}^j \frac{\partial}{\partial y^j}$$

from the x^i to the y^i coordinates. From the lemma we find

$$\tilde{v}^j = \frac{\partial x^i}{\partial y^j} v^i$$

and similarly for a covector field.
One can also easily recover the components v^i from the vector field v by evaluating v on the coordinate functions:

$$v^i = v(x^i)$$

and similarly for a covector field ω:

$$\omega_i = \omega\left(\frac{\partial}{\partial x^i}\right)$$

We introduce the differential of a map, which can be interpreted as its linearisation at a point.

Definition 2.16 (Differential / pushforward of a Map). Let $F : M \to N$ be a smooth map. The push forward or differential F_* or $dF : T_pM \to T_{F(p)}N$ is defined by

$$dF(v)(h) = v(h \circ F).$$

In local coordinates this is the Jacobian matrix of F, $[dF]^a_b = \frac{\partial F^a}{\partial x^b}$. Equivalently, if $\gamma : (-\epsilon, \epsilon) \to M$ is a curve with $\gamma(0) = p$ and $\dot{\gamma}(0) = v$, then

$$dF_p(\dot{\gamma}(0)) = \frac{d}{dt}(F \circ \gamma)(t)\bigg|_{t=0}.$$

Definition 2.17 (Immersions, Embeddings, and Diffeomorphisms). We introduce three important types of maps, based on the rank of the differential dF.

- F is an immersion if dF is injective.
- F is an embedding if it is an immersion and a homeomorphism onto its image.
- F is a diffeomorphism if it is bijective and both F and F^{-1} are smooth.

Theorem 2.18 (Inverse Function Theorem). If $F : M \to N$ is smooth and $dF : T_pM \to T_{F(p)}N$ is
- injective $\implies$ locally around p F is an embedding

- bijective $\implies$ locally around p F is a diffeomorphism

Proof. See Theorems 4.8 and 4.14 in [TR24]. □

3 Riemannian metrics

(Lecture 2)

Definition 3.1 (Riemannian Metric). A Riemannian metric g on a manifold M of dimension n is a map $g_p : T_pM \times T_pM \to \mathbb{R}$ at every point $p \in M$ such that:

1. g_p is bilinear: $g_p(\alpha w + \beta u, v) = \alpha g_p(w, v) + \beta g_p(u, v)$ for all $\alpha, \beta \in \mathbb{R}$ and $u, v, w \in T_pM$
2. g_p is symmetric: $g_p(w, v) = g_p(v, w)$
3. g_p is positive definite: $g_p(v, v) \geq 0$ with equality iff $v = 0$
4. g_p is smooth: $p \mapsto g_p$ is smooth in any local coordinate system (see below).

We will sometimes use the notation $\langle v, w \rangle := g(v, w)$ for the metric.
g is a section of the bundle of covariant 2-tensors on M. Therefore it can be written in local coordinates as

$$g = g_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j \tag{3.1}$$

and the components g_{ij} can be found by

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right).$$

This is what the smoothness condition means: g_{ij} are smooth functions. And in our convention, they live on the manifold M. The matrix of components $[g_{ij}]$ is symmetric, positive definite. We would like to remark, that some people also omit the $\otimes$ symbol and write $g = g_{ij} \mathrm{d}x^i \mathrm{d}x^j$.

Lemma 3.2 (Change of coordinates for metric components).

$$\tilde{g}_{ij} \frac{\partial y^i}{\partial x^a} \frac{\partial y^j}{\partial x^b} = g_{ab}$$

Proof. See exercise 1.3. □

In matrix form:

$$[\tilde{g}] = \left[\frac{\partial x}{\partial y}\right]^T [g] \left[\frac{\partial x}{\partial y}\right]$$

Definition 3.3 (Euclidean Metric). Let $M = \mathbb{R}^n$, then we define the Euclidean metric (or *flat* metric) g_E (or $g_{\mathbb{R}^n}$) to coincide with the Euclidean inner product on $\mathbb{R}^n$:

$$g_E(v, w) := \langle v, w \rangle_{\mathbb{R}^n}.$$

This means that its components are given by

$$\begin{aligned} g_{ij} &= g_E\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \\ &= g_E\left(0\frac{\partial}{\partial x^1} + \ldots + 1\frac{\partial}{\partial x^i} + \ldots 0\frac{\partial}{\partial x^n}, 0\frac{\partial}{\partial x^1} + \ldots + 1\frac{\partial}{\partial x^j} + \ldots 0\frac{\partial}{\partial x^n}\right) \\ &= \delta_{ij} \end{aligned}$$

Expanded in the Euclidean standard basis, we have

$$g_E = \mathrm{d}x^i \otimes \mathrm{d}x^i$$

or also written as $(\mathrm{d}x^1)^2 + \cdots + (\mathrm{d}x^n)^2$.
When expanding the same metric in polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ (on $\mathbb{R}^2$), we get

$$\tilde{g}_E = \mathrm{d}r \otimes \mathrm{d}r + r^2 \mathrm{d}\theta \otimes \mathrm{d}\theta.$$

See exercise 1.3 for a derivation.

We make the following informal but nevertheless useful definition.

Definition 3.4 (Geometric property). A geometric property is a property that is invariant under changes of coordinates.

More concretely: objects in M or T_pM (or products and sections thereof) that can be described in different coordinates may have such properties.
Examples of such properties are

- Smoothness of a function, (co)vector field, or metric.
- Norm of a tangent vector
$$\|x\| := \sqrt{g(x,x)}$$
- Angle $\angle$ between two non-zero vectors
$$\cos(\angle(u,v)) := \frac{g(u,v)}{\|u\| \cdot \|v\|}$$
- Length of a curve (see lemma 3.6)

3.1 Lengths and distances

A curve $\gamma : (a,b) \to M$ is called piecewise C^k if it is continuous and there exist $t_0 = a < t_1 < \cdots < t_m = b$ such that

$$\gamma|_{[t_i,t_{i+1}]} \text{ is } C^k \quad \forall i = 0,\ldots,m-1.$$

Definition 3.5 (Length of a curve). Let $\gamma : (a,b) \to (M,g)$ be piecewise C^1. Then the length of γ is

$$l(\gamma) = \int_a^b \|\dot{\gamma}(t)\|\, dt$$

Technically to define the length, absolute continuity, or $\gamma \in W^{1,1}(a,b)$ would be enough.

Lemma 3.6 (Characteristics of the length functional). The length functional has the following properties:

1. $l(\gamma) > 0$, and "=" iff γ is constant.
2. Additivity: If $\gamma : [a,b] \to M$, $c \in (a,b)$:
$$l(\gamma|_{[a,b]}) = l(\gamma|_{[a,c]}) + l(\gamma|_{[c,b]})$$
3. Invariant under reparametrization: If $h : (c,d) \to (a,b)$ is piecewise C^1, monotone and

onto, then

$$l(\gamma \circ h) = l(\gamma)$$

Property 3 shows that the length is a geometric quantity, independent of the parametrization.

Proof. See exercise 2.1. □

Definition 3.7 (Riemannian Distance Function). Let (M, g) be a Riemannian manifold; $p, q \in M$. We define the Riemannian distance function:

$$\begin{aligned} d_g : M \times M &\to [0, +\infty] \\ (p, q) &\mapsto \inf \left\{ l(\gamma), \gamma \text{ piecewise } C^1, \gamma(0) = p, \gamma(1) = q \right\} \end{aligned}$$

The Riemannian distance has the following properties.

Lemma 3.8. The distance function satisfies $d_g(p, q) = +\infty$ iff p, q belong to different connected components of M.
Further, (M, d_g) is a metric space, i.e.

- $d_g(p, q) = d_g(q, p)$
- $d_g(p, q) \leq d_g(p, z) + d_g(z, q)$
- $d_g(p, q) \geq 0$
- $d_g(p, q) = 0 \iff p = q$

Proof. See exercise 2.2. □

3.2 Integration on Riemannian manifolds

Let (M, g) be a Riemannian manifold and $h \in C^0(M)$. We want to define the integral:

$$\int_M h$$

This itself makes no sense a priori, as there is no natural notion of integration of functions on manifolds. One can only meaningfully integrate differential forms over manifolds. We will therefore use the metric to define a volume form, which we can use to integrate functions.

Definition 3.9 (Integral over a chart domain). Let (U, ϕ) be a local coordinate chart, with associated coordinates $(x^1, \ldots, x^n)$. We define

$$\int_U h \, dV_g := \int_{\phi(U)} h\left(\phi^{-1}(x)\right) \sqrt{\det\left[g_{ij}(x)\right]} \circ \phi^{-1} \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$$

The wedge-product integral on the right-hand side is the usual integral on $\mathbb{R}^n$, i.e.

$$\int_{\phi(U)} f \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n := \int_{\phi(U)} f \, \mathrm{d}x^1 \cdots \mathrm{d}x^n,$$

as explained in Definition 11.11 in [TR24].
Here, $h \circ \phi^{-1}$ is the coordinate representation of h, and $\sqrt{\det[g_{ij}]}$ is the volume density.

The volume form of (M, g) is the top-form dV_g expressed in local coordinates by

$$dV_g = \sqrt{\det[g_{ij}]} \circ \phi^{-1} \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n.$$

Integrating hdV_g over U is well defined, see Definition 11.11 in [TR24].
Attention: the d in dV_g does not mean this is a differential. It is an abuse of notation make the integral look like to usual one on $\mathbb{R}^n$.
Furthermore we want to remark that the above definition assumes that g_{ij} are functions on M, not $\mathbb{R}^n$.

We can now glue the charts together to get an integral over M.

Definition 3.10 (Integral over M). Let $\{\chi_a\}_{a\in A}$ be a partition of unity, such that $\operatorname{supp} \chi_a$ is contained in some local coordinate chart U_a. Then we define

$$\int_M h \, dV_g := \sum_{a \in A} \int_U h \cdot \chi_a \, dV_g$$

Lemma 3.11. The above definitions are coordinate independent.

Proof. See exerise 1.5, and chapter 11 in [TR24]. □

As functions, metrics can be pulled back via smooth maps, which allows to define new metrics from old ones.

Definition 3.12 (Pullback of a Metric). Let $F : M \to (N, g)$ be an immersion. We define $\forall p \in M, \forall x, y \in T_pM$ the pullback of g via F by

$$F^*g(x, y) := g(\mathrm{d}F_p(x), \mathrm{d}F_p(y))$$

We also say that F^*g is the metric induced on M by F.

Lemma 3.13. F^*g is a Riemannian metric.

Proof. It is obviously symmetric and bilinear. Positivity (using that F is an immersion): $F^*g(x, x) = g(\mathrm{d}F(x), \mathrm{d}F(x)) > 0$, "=" iff $\mathrm{d}F(x) = 0 \iff x = 0$. □

In local coordinates one gets

$$F^*g_{ij} = g\left(\mathrm{d}F\left(\frac{\partial}{\partial x^i}\right), \mathrm{d}F\left(\frac{\partial}{\partial x^j}\right)\right) = \frac{\partial F^a}{\partial x^i} g_{ab} \frac{\partial F^b}{\partial x^j}.$$

In exercise 1.4, we derive this for the case where g is the Euclidean metric.

In exercise 2.3, we pull back the Euclidean metric to the sphere via the stereographic projection and get the so called *round metric*. This exercise also shows that sometimes, when pulling back the Euclidean metric in particular, it is easier to use the fact that

$$g_E = dx^i \otimes dx^i$$

together with the fact that the pullback of a tensor product is the tensor product of the pullbacks, i.e. in this case $F^*g_E = F^*dx^i \otimes F^*dx^i$. This is a very straight forward generalisation of Lemma 8.19)b) in [TR24].

Example 3.14. Let M be the graph of a function:

$$f : \mathbb{R}^{n-1} \to \mathbb{R} \quad \text{in } \mathbb{R}^n.$$

In local coordinates $(x^1, \ldots, x^{n-1})$ on M, f induces an immersion

$$F(x^1, \ldots, x^{n-1}) = (x^1, \ldots, x^{n-1}, f(x^1, \ldots, x^{n-1})).$$

So we have

$$\frac{\partial F}{\partial x^i} = (0, \ldots, 0, 1, 0, \ldots, 0, \frac{\partial f}{\partial x^i}).$$

This gives us the following induced metric on M:

$$(F^* g_E)_{ij} = g_E\left(\frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j}\right) = \delta_{ij} + \frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}.$$

3.3 Isometries and musical isomorphisms

We recall the following *inner-product preserving* map on linear spaces.

Definition 3.15 (Linear Isometry on Tangent Spaces). Let T_pM and T_qN be tangent spaces equipped with metrics g and h, respectively. A linear map $L : T_pM \to T_qN$ is a *linear isometry* if

$$h(Lv, Lw) = g(v, w) \quad \text{for all } v, w \in T_pM.$$

With this, we can define *metric preserving* maps between Riemannian manifolds.

Definition 3.16 (Isometries). Let $F : (M, g) \to (N, h)$ be a smooth map.

- F is a local isometry if $\mathrm{d}F_p : (T_pM, g_p) \to (T_{F(p)}N, h)$ is a linear isometry for all $p \in M$.
- F is a global isometry if it is a local isometry and bijective. In this case (M, g) and (N, h) are geometrically identical.
- F is locally conformal if $\mathrm{d}F_p$ is bijective and $F^*h = \lambda g$ for some $\lambda \in C^\infty(M), \lambda > 0$.

Local isometries (i.e. their differentials) preserve norms and angles, locally conformal maps (i.e. their differentials) preserve angles.
F being a local isometry implies that $\mathrm{d}F_p$ is bijective and hence F is an immersion and hence a local diffeomorphism.
If $\mathrm{d}F_p$ is a linear isometry for all $p \in M$, then

$$(F^*h)_p(v, w) = h_{F(p)}\left(\mathrm{d}F_p v, \mathrm{d}F_p w\right) = g_p(v, w) \quad \text{for all } v, w \in T_pM.$$

Hence,

$$F^*h = g.$$

Lemma 3.17. If $F : (M, g) \to (N, h)$ is an isometry of metric spaces, then F is smooth and a Riemannian isometry.

Proof. Was skipped in the lecture. □

Theorem 3.18. If M is a smooth manifold, then it admits a Riemannian metric.

Proof. Let $\{U_a, \phi_a\}_{a\in A}$ be an atlas on M and let $\{\chi_\beta\}_{\beta\in B}$ be a subordinate partition of unity, i.e. $\forall \beta \in B : \chi_\beta \in C^\infty(M, [0,1])$, $\operatorname{supp}\chi_\beta \subseteq U_{a(\beta)}$ for some $a(\beta) \in A$, $\operatorname{supp}\chi_\beta \cap \operatorname{supp}\chi_{\beta'} \neq \emptyset$ for finitely many β, $\sum_\beta \chi_\beta = 1$.
For all $\beta \in B$: On $U_{a(\beta)}$ consider the metric $g_\beta = \phi^*_{a(\beta)} g_E$.
Set $g = \sum_\beta \chi_\beta \cdot g_\beta$.
Then, g is well-defined (finite sum at every point). In addition, g is symmetric, bilinear. What is left is to check positive definiteness. We have

$$g(x,x) > 0 \quad \forall x \in T_pM$$

and further

$$g(x,x) = 0 \iff \sum_\beta \chi_\beta \cdot g_\beta(x,x) = 0$$

which implies that for all β

$$\chi_\beta \cdot g_\beta(x,x) = 0.$$

And this means that if $\chi_\beta(p) \neq 0$ for some β, then

$$g_\beta(x,x) = 0 \implies x = 0.$$

□

Lemma 3.19 (Musical Isomorphism). Let (M,g) be a Riemannian manifold. For any 1-form $\omega \in \Gamma(T^*M)$, define the vector field $\omega^\# \in \Gamma(TM)$ by the condition

$$g(X, \omega^\#) = \omega(X) \quad \forall X \in \Gamma(TM).$$

Due to positive definiteness of g, this vector field is uniquely defined. Then we have in local coordinates

$$(\omega^\#)^i = g^{ij}\omega_j,$$

where (g^{ij}) is the inverse of the metric matrix (g_{ij}).

Proof. See exercise 3.4a. □

Definition 3.20 (Gradient of a Function). Let (M,g) be a Riemannian manifold and $f \in C^\infty(M)$. The gradient $\nabla f \in \Gamma(TM)$ is defined by $g(\nabla f, X) = \mathrm{d}f(X)$ for all $X \in \Gamma(TM)$, or equivalently

$$\nabla f := (\mathrm{d}f)^\#.$$

In local coordinates, $\nabla f = g^{ij}\frac{\partial f}{\partial x^j}\frac{\partial}{\partial x^i}$, where (g^{ij}) is the inverse of the metric matrix (g_{ij}). See also exercise 3.4b.

Lemma 3.21 (Induced Metric on Cotangent Space). Let (M,g) be a Riemannian manifold. Define a $(2,0)$-tensor $\tilde{g}$ on 1-forms by

$$\tilde{g}(\omega_1, \omega_2) := g(\omega_1^\#, \omega_2^\#) \quad \forall \omega_1, \omega_2 \in \Gamma(T^*M).$$

Then $\tilde{g}$ is symmetric and positive definite. In local coordinates, its coefficients are

$$\tilde{g}^{ij} = g^{ij},$$

where g^{ij} is the inverse matrix of the metric coefficients g_{ij}.

Proof. See exercise 3.4c. □

4 Tensors

(Lecture 3)

Definition 4.1 (Tangent bundle). The tangent bundle is the disjoint union of all tangent spaces of M,

$$TM = \bigsqcup_{p \in M} T_p M$$

and admits a natural projection map $\pi : TM \to M$, $(p, v) \mapsto p$

Lemma 4.2 (Properties of TM). It naturally has the structure of a smooth manifold of dimension $\dim(TM) = 2\dim(M)$.
If (U, ϕ) is a local coordinate chart on M with coordinates $(x^1, \dots, x^n)$, then the tangent bundle $TU = \bigsqcup_{p \in U} T_p M$ inherits a coordinate chart defined by

$$\begin{aligned} \widetilde{\phi} : TU &\longrightarrow \mathbb{R}^{2n} \\ (p, v) &\mapsto (x^1(p), \dots, x^n(p), v^1, \dots, v^n) \end{aligned}$$

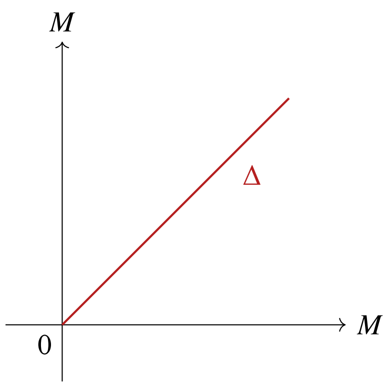
where $v = v^i \frac{\partial}{\partial x^i}|_p$ are the components of v in the coordinate basis at p.
Interestingly, one can show that any section, i.e. any vector field $X : M \to TM$, is an immersion!

Proof. See chapter 3 of [TR24] and exercise 3.2. □

So locally: TM looks like the product $U \times \mathbb{R}^n$. But not necessarily globally, see example below.

Example 4.3. $M = \mathbb{S}^1$, then $T\mathbb{S}^1 \cong \mathbb{S}^1 \times \mathbb{R}$ (the tangent bundle is trivial). But $T\mathbb{S}^2$ is not a product bundle (it is a non-trivial vector bundle).

Remark 4.4. As a manifold, TM is always homeomorphic to an open neighborhood of the diagonal Δ in $M \times M$.



(Given a Riemannian metric, this identification follows using the exponential map, introduced later.)
So for any two differentiable structures on M, the corresponding tangent bundles are homeomorphic as manifolds and they have the same homotopy invariants.

Definition 4.5 (Vector Field). A vector field X is a smooth section (=right inverse) of the projection map π of the tangent bundle TM, i.e., a smooth map

$$X : M \longrightarrow TM, \quad p \longrightarrow X_p \in T_pM.$$

In other words given the bundle projection $\pi : TM \to M$, we have $\pi \circ X = \mathrm{id}_M$.
Recall, that the set of all vector fields on M is denoted by $\Gamma(TM)$ (or sometimes by $\Gamma(M)$ or $\mathfrak{X}(M)$).

$\Gamma(TM)$ is a $C^\infty(M)$ module. If $f \in C^\infty(M)$, $X \in \Gamma(TM)$, then $fX \in \Gamma(TM)$. This means that $\Gamma(TM)$ is a module over the ring $C^\infty(M)$: we can multiply vector fields by smooth functions and add them together, and these operations satisfy the expected properties. This structure is fundamental to differential geometry, as it allows us to work with vector fields as objects that can be scaled by functions and combined linearly.

Similarly for T^*M, we can define the cotangent bundle.

Definition 4.6 (Cotangent Bundle). The cotangent bundle is the disjoint union of all cotangent spaces of M,

$$T^*M = \bigsqcup_{p \in M} T_p^*M$$

and admits a natural projection map $\pi : T^*M \to M$, $(p, \omega) \mapsto p$.

As for TM, it has a natural smooth manifold structure of dimension $2\dim(M)$ and locally looks like $U \times \mathbb{R}^n$, with charts

$$(p, \omega) \mapsto (x^1(p), \dots, x^n(p), \omega_1, \dots, \omega_n),$$

where $\omega = \omega_i \mathrm{d}x^i|_p$.

Definition 4.7 (1-forms). A 1-form is a smooth section of the cotangent bundle T^*M, i.e., a smooth map $\omega : M \to T^*M, p \mapsto \omega_p$. The set of all 1-forms on M is denoted by $\Gamma(T^*M)$ (or $\Gamma^*(M)$).

Example 4.8. If $f_1, \dots, f_s, h_1, \dots, h_s \in C^\infty(M)$,

$$\omega = \sum_{i=1}^{s} f_i \, \mathrm{d}h_i$$

is a 1-form.

Remark 4.9 (Straightening vector fields). If $X \in \Gamma(TM)$, $X|_p \neq 0$: Locally one can find coordinates around p such that $X = \frac{\partial}{\partial x^1}$.
However, if $\omega \in \Gamma(T^*M)$, then in general, one cannot express ω as $\mathrm{d}f$ for some $f \in C^\infty(M)$ (unless $\mathrm{d}\omega = 0$).

4.1 Tensors at a point p

Let $\omega \in T_p^*M$. I can think of it as a linear map

$$\omega : T_pM \longrightarrow \mathbb{R}.$$

Similarly: I can think of any $v \in T_pM$ as

$$v : T_p^*M \longrightarrow \mathbb{R}$$

Definition 4.10 (Tensor at a point). A tensor of type (k, l) at p is a multilinear map

$$T : \underbrace{T_p^*M \times \cdots \times T_p^*M}_{k \text{ times}} \times \underbrace{T_pM \times \cdots \times T_pM}_{l \text{ times}} \longrightarrow \mathbb{R}.$$

Remark 4.11 (General tensors). One can easily generalize tensors to multilinear maps on general finite-dimensional vectorspaces and their duals, independent of any manifold structure. Then one considers multilinear maps of the form

$$T : \underbrace{V^* \times \cdots \times V^*}_{k \text{ times}} \times \underbrace{V \times \cdots \times V}_{l \text{ times}} \longrightarrow \mathbb{R}.$$

Example 4.12. Some important examples of tensors to develop intuition.

- Tangent vector $v \in T_pM$: $(1, 0)$ tensor
- Covector $\omega \in T_p^*M$: $(0, 1)$ tensor
- Riemannian metric $g \in T_p^*M \otimes T_p^*M$: $(0, 2)$ tensor

The first index denotes the *vectorial* part, the second one the *covectorial* part (nonstandard terminology).

4.2 Tensor product

Definition 4.13 (Tensor Product). Given T_1 of type (k_1, l_1) and T_2 is of type (k_2, l_2), then $T_1 \otimes T_2$ is a tensor of type $(k_1 + k_2, l_1 + l_2)$ defined by

$$\begin{aligned}(T_1 \otimes T_2)&(\omega_1, \ldots, \omega_{k_1+k_2}; v_1, \ldots, v_{l_1+l_2}) \\ &:= T_1(\omega_1, \ldots, \omega_{k_1}; v_1, \ldots, v_{l_1}) \cdot T_2(\omega_{k_1+1}, \ldots, \omega_{k_1+k_2}; v_{l_1+1}, \ldots, v_{l_1+l_2})\end{aligned}$$

One can easily check the following facts:

- The tensor product is not commutative: $T_1 \otimes T_2 \neq T_2 \otimes T_1$.
- It is bilinear in each argument:

$$T_1 \otimes (aT_2) = a \cdot (T_1 \otimes T_2), \quad a \in \mathbb{R}$$
$$T_1 \otimes (T_2 + T_3) = T_1 \otimes T_2 + T_1 \otimes T_3$$

4.3 Coordinate basis

Recall from lemmas 2.10 and 2.12 if $(x^1, \ldots, x^n)$ local coordinates around p then

$$\left\{\frac{\partial}{\partial x^i}\right\}_{i=1}^n = \text{Basis of } T_pM$$
$$\left\{\mathrm{d}x^i\right\}_{i=1}^n = \text{Basis of } T_p^*M$$

Lemma 4.14 (Basis of (k,l)-tensors). The set

$$\left\{\frac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_k}}\otimes \mathrm{d}x^{j_1}\otimes\cdots\otimes\mathrm{d}x^{j_l}\right\}_{(i_1,\dots,i_k,j_1,\dots,j_l)}$$

forms a basis of the space of (k,l)-tensors at p.

Proof. See Proposition C.9 in [TR24]. □

Because of the above lemma, for every tensor T there exist unique components $T^{i_1\cdots i_k}_{j_1\cdots j_l}$ such that

$$T = T^{i_1\cdots i_k}_{j_1\cdots j_l}\frac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_k}}\otimes \mathrm{d}x^{j_1}\otimes\cdots\otimes\mathrm{d}x^{j_l}$$

That's why we have been writing $g = g_{ij}\,\mathrm{d}x^i\otimes\mathrm{d}x^j$, see equation (3.1).

We use indices up for the contravariant (vectorial) indices and indices down for the covariant (covectorial) indices.

To recover the components of a tensor T in local coordinates, we can apply T to the corresponding basis vectors, see Proposition C.9 in [TR24].. For example, for a (k,l)-tensor T, we have

$$T^{i_1\cdots i_k}_{j_1\cdots j_l} = T\left(\mathrm{d}x^{i_1},\dots,\mathrm{d}x^{i_k};\frac{\partial}{\partial x^{j_1}},\dots,\frac{\partial}{\partial x^{j_l}}\right). \tag{4.1}$$

4.4 Change of coordinates

If we want to switch from one coordinate system to another, we would like to know how the components of a tensor change under a change of coordinates. Let us consider a change of coordinates

$$(x^1,\dots,x^n)\longrightarrow(y^1,\dots,y^n)$$

Recall from lemma 2.14 that $\mathrm{d}y^i = \frac{\partial y^i}{\partial x^a}\cdot\mathrm{d}x^a$ and $\frac{\partial}{\partial y^j} = \frac{\partial x^b}{\partial y^j}\cdot\frac{\partial}{\partial x^b}$. Then we have the following lemma.

Lemma 4.15 (Change of coordinates for tensors). Under a change of coordinates $(x^1,\dots,x^n)\longrightarrow(y^1,\dots,y^n)$, the components of a (k,l)-tensor transform as:

$$T^{i_1\dots i_k}_{j_1\dots j_l} = T^{a_1\dots a_k}_{b_1\dots b_l}\frac{\partial y^{i_1}}{\partial x^{a_1}}\cdots\frac{\partial y^{i_k}}{\partial x^{a_k}}\cdot\frac{\partial x^{b_1}}{\partial y^{j_1}}\cdots\frac{\partial x^{b_l}}{\partial y^{j_l}}$$

Proof. Let T be a (k,l)-tensor at p and recall equation (4.1)

$$T^{a_1\dots a_k}_{b_1\dots b_l} = T(\mathrm{d}x^{a_1},\dots,\mathrm{d}x^{a_k};\frac{\partial}{\partial x^{b_1}},\dots,\frac{\partial}{\partial x^{b_l}}).$$

Under a change of coordinates $x\mapsto y$, the basis transforms as

$$\mathrm{d}y^i = \frac{\partial y^i}{\partial x^a}\mathrm{d}x^a,\qquad \frac{\partial}{\partial y^j} = \frac{\partial x^b}{\partial y^j}\frac{\partial}{\partial x^b}.$$

Applying T to the new basis gives

$$\begin{aligned}
T(\mathrm{d}y^{i_1},\dots,\mathrm{d}y^{i_k};\frac{\partial}{\partial y^{j_1}},\dots,\frac{\partial}{\partial y^{j_l}}) &= T\Big(\frac{\partial y^{i_1}}{\partial x^{a_1}}\mathrm{d}x^{a_1},\dots,\frac{\partial y^{i_k}}{\partial x^{a_k}}\mathrm{d}x^{a_k};\frac{\partial x^{b_1}}{\partial y^{j_1}}\frac{\partial}{\partial x^{b_1}},\dots,\frac{\partial x^{b_l}}{\partial y^{j_l}}\frac{\partial}{\partial x^{b_l}}\Big)\\
&= \frac{\partial y^{i_1}}{\partial x^{a_1}}\cdots\frac{\partial y^{i_k}}{\partial x^{a_k}}\frac{\partial x^{b_1}}{\partial y^{j_1}}\cdots\frac{\partial x^{b_l}}{\partial y^{j_l}}T(\mathrm{d}x^{a_1},\dots,\mathrm{d}x^{a_k};\frac{\partial}{\partial x^{b_1}},\dots,\frac{\partial}{\partial x^{b_l}})\\
&= \frac{\partial y^{i_1}}{\partial x^{a_1}}\cdots\frac{\partial y^{i_k}}{\partial x^{a_k}}\frac{\partial x^{b_1}}{\partial y^{j_1}}\cdots\frac{\partial x^{b_l}}{\partial y^{j_l}}T^{a_1\dots a_k}_{b_1\dots b_l}.
\end{aligned}$$

□

This transformation behavior is what historically led to the following terminology for the indices of a tensor:

- Covariant part: Indices down (transforms with multiplication with the Jacobian for the y-to-x *transformation* $\frac{\partial x}{\partial y}$)
- Contravariant part: Indices up (transforms with inverse Jacobian $\frac{\partial y}{\partial x}$)

Definition 4.16 (Space of Tensors). The space of tensors of type (k,l) at a point $p \in M$ is denoted by

$$\otimes^k T_p M \otimes^l T_p^* M.$$

By Definition 4.10, its elements are the multilinear maps

$$Q : \underbrace{T_p^* M \times \cdots \times T_p^* M}_{k \text{ times}} \times \underbrace{T_p M \times \cdots \times T_p M}_{l \text{ times}} \longrightarrow \mathbb{R}.$$

Remark 4.17 (Characterisation of tensors). Recall Remark 4.11. Let T be a (k,l) tensor over V, meaning

$$T : \underbrace{V^* \times \cdots \times V^*}_{k \text{ times}} \times \underbrace{V \times \cdots \times V}_{l \text{ times}} \longrightarrow \mathbb{R}.$$

Then, by forgetting (k_1, l_1) of its entries, we can think of it as a multilinear map in the rest of the (k_1, l_1)-variables. This way we get

$$T : \underbrace{V^* \times \cdots \times V^*}_{k-k_1} \times \underbrace{V \times \cdots \times V}_{l-l_1} \longrightarrow \otimes^{k_1} V \otimes^{l_1} V^*,$$

so the image of T is a multilinear map on the rest of the (k_1, l_1)-variables.

Consider as a simple example, the $(1,1)$ tensor Q (so $Q \in \otimes^1 V \otimes^1 V^* = V \otimes V^*$). Then Q is a bilinear map

$$Q : V^* \times V \to \mathbb{R}.$$

If I forget the first variable, I can think of it as a map $Q : V \to V$ defined by

$$Q : V \to V, v \mapsto Q(\cdot, v).$$

Similarly, if I forget the second variable, I can think of it as a map

$$Q : V^* \to V^*, \omega \mapsto Q(\omega, \cdot).$$

Obviously *forgetting* means to evaluate the tensor at this variable, and then considering the remaining variables as the input of the resulting multilinear map.
We remark that here we have identified $V \cong (V^*)^*$.

Lemma 4.18 (Tensor Product in Coordinates). Let f be a (k_1, l_1)-tensor and g be a (k_2, l_2)-tensor with components $f^{j_1 \cdots j_{l_1}}_{i_1 \cdots i_{k_1}}$ and $g^{j_1 \cdots j_{l_2}}_{i_1 \cdots i_{k_2}}$ respectively. Then the tensor product $f \otimes g$ is a $(k_1 + k_2, l_1 + l_2)$-tensor with components

$$(f \otimes g)^{j_1 \cdots j_{l_1} j_{l_1+1} \cdots j_{l_1+l_2}}_{i_1 \cdots i_{k_1} i_{k_1+1} \cdots i_{k_1+k_2}} = f^{j_1 \cdots j_{l_1}}_{i_1 \cdots i_{k_1}} \cdot g^{j_{l_1+1} \cdots j_{l_1+l_2}}_{i_{k_1+1} \cdots i_{k_1+k_2}}.$$

We introduce another operation on tensors: the contraction. Given a tensor of type (k, l), with components $T^{i_1 \cdots i_k}_{j_1 \cdots j_l}$, we can contract one contravariant and one covariant index to get a tensor of type $(k-1, l-1)$ with components

Definition 4.19 (Contraction of Tensors in coordinates). Choose one contravariant and one covariant index and sum them:

$$\operatorname{tr}_{(k,l)} : \{(k,l) \text{ tensors}\} \longrightarrow \{(k-1, l-1) \text{ tensors}\}$$

with respect to the i-th index up, j-th index down by the rule

$$(\operatorname{tr}_{(i_a, j_b)} T)^{i_1 \cdots i_{a-1} i_{a+1} \cdots i_k}_{j_1 \cdots j_{b-1} j_{b+1} \cdots j_l} = T^{i_1 \cdots i_{a-1} a i_{a+1} \cdots i_k}_{j_1 \cdots j_{b-1} a j_{b+1} \cdots j_l},$$

where we sum over the repeated index a as always.

Example 4.20. To understand this, consider $a = b = 1$. Then we have

$$(\operatorname{tr}_{(1,1)} T)^{i_2 \cdots i_k}_{j_2 \cdots j_l} = T^{a i_2 \cdots i_k}_{a j_2 \cdots j_l}.$$

In the case of a $(1,1)$ tensor: This is just its trace as a linear map. Indeed, if Q is a $(1,1)$ tensor, we can identify it with a linear map $Q : V \to V$ and $\operatorname{tr}_{(1,1)} Q = Q^i_i$ is the usual trace of this linear map.

We remark, that there is a more abstract, but *less useful* definition that does not rely on coordinates.

As the usual trace of a linear map, the above definition is in fact independent of the choice of coordinates.

Lemma 4.21 (Contraction is coordinate-independent). The contraction of tensors is independent of the choice of coordinates.

Proof. See exercise 3.1. □

4.5 Tensor fields

Definition 4.22 (Tensor field, space of tensor fields). A tensor field of type (k, l) on M is an assignment at every point $p \in M$ of a (k, l) tensor T_p, such that the components $T^{i_1 \cdots i_k}_{j_1 \cdots j_l}$ are smooth functions in any local coordinate system.
We will denote the space of all (k, l)-tensor fields by

$$\Gamma(\otimes^k TM \otimes^l T^*M).$$

For brevity, the following notation is also common:

$$T^k_l(M) := \Gamma(\otimes^k TM \otimes^l T^*M).$$

See also [TR24], chapter 8.2.

Of course for $T \in \Gamma(\otimes^k TM \otimes^l T^*M)$ we have that for any $p \in M$, $T_p \in \otimes^k T_pM \otimes^l T^*_pM$ is a (k, l)-tensor at p, according to Definition 4.16.

The Riemannian metric is an example of a smooth tensor field of type $(0, 2)$.

Remark 4.23. If T is a (k,l)-tensor field, we can view it as a $C^\infty(M)$-multilinear map

$$T : \underbrace{\Gamma(T^*M) \times \cdots \times \Gamma(T^*M)}_{k \text{ times}} \times \underbrace{\Gamma(TM) \times \cdots \times \Gamma(TM)}_{l \text{ times}} \longrightarrow C^\infty(M).$$

The key property is that T is *pointwise*: for any $\omega_1, \ldots, \omega_k \in \Gamma(T^*M)$ and $X_1, \ldots, X_l \in \Gamma(TM)$, the value $T(\omega_1, \ldots, \omega_k, X_1, \ldots, X_l)$ at a point $p \in M$ depends only on the values of these fields at p:

$$T(\omega_1, \ldots, \omega_k, X_1, \ldots, X_l)|_p = T_p(\omega_1|_p, \ldots, \omega_k|_p, X_1|_p, \ldots, X_l|_p).$$

In particular, if any of the vector field arguments vanishes at p (i.e., $X_i|_p = 0$), then the result is zero at p.
This pointwise property is equivalent to T being $C^\infty(M)$-multilinear in its arguments.

Example 4.24 (Divergence is not a tensor). Consider the divergence operator $\mathrm{div} : \Gamma(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ defined by

$$\mathrm{div}(X) = \frac{\partial X^i}{\partial x^i}.$$

This is not a tensor field because it fails to be $C^\infty(\mathbb{R}^n)$-multilinear. Specifically

$$\mathrm{div}(fX) = \frac{\partial(fX^i)}{\partial x^i} = f\frac{\partial X^i}{\partial x^i} + X^i\frac{\partial f}{\partial x^i} \neq f \cdot \mathrm{div}(X).$$

This violates the required linearity property.

The example indicates that for T to be a tensor field, it must be $C^\infty(M)$-multilinear in its arguments. However we get more.

Theorem 4.25 (Characterization of Tensor Fields). A map

$$T : \underbrace{\Gamma(T^*M) \times \cdots \times \Gamma(T^*M) \times \Gamma(TM) \times \cdots \times \Gamma(TM)}_{k \text{ copies of } \Gamma^*,\ l \text{ copies of } \Gamma} \longrightarrow C^\infty(M)$$

is a tensor field according to definition 4.22 if and only if it is $C^\infty(M)$-multilinear.

Proof idea. The direction "tensors are $C^\infty(M)$-multilinear" is true by definition. The converse, "$C^\infty(M)$-linear functional $\implies$ 1-form" is harder and needs a proof. For this one first shows that $\omega(Y)$ at p depends only on $Y|_p$, i.e., it acts pointwise. Why? Because of what a tensor field is: it is an assignment $p \mapsto T_p$, where T_p is a tensor that only depends on p.

For (0,1) tensors, see exercise 3.5. □

4.6 Notation summary of bundles, sections and tensors

We have introduced many different bundles and their spaces of sections. For convenience, we summarize them in the following table.
The idea is always as follows: The space $\Gamma(E)$ denotes all smooth sections of a vector bundle $E \to M$, and the symbol Λ denotes bundles of alternating tensors (forms and multivectors).
For completeness, we also include bivector and k-vector fields, which are not central to this course (we mention them in Remarks 8.10 and 8.20). They are defined analogously to k-forms (i.e. they are alternating tensors), but with the tangent bundle instead of the cotangent bundle.

Object	Bundle	Space of sections	Shorthand notations
Vector fields	TM	$\Gamma(TM)$	$\mathfrak{X}(M)$
1-forms	T^*M	$\Gamma(T^*M)$	$\mathfrak{X}^*(M)$ or $\Omega^1(M)$
Bivector fields	$\Lambda^2 TM$	$\Gamma(\Lambda^2 TM)$	-
2-forms	$\Lambda^2 T^*M$	$\Gamma(\Lambda^2 T^*M)$	$\Omega^2(M)$
k-vector fields	$\Lambda^k TM$	$\Gamma(\Lambda^k TM)$	-
k-forms	$\Lambda^k T^*M$	$\Gamma(\Lambda^k T^*M)$	$\Omega^k(M)$
(k,l)-tensor fields	$\otimes^k TM \otimes^l T^*M$	$\Gamma(\otimes^k TM \otimes^l T^*M)$	$T^k_l(M)$

Some authors use $T^k(TM)$ instead of $\otimes^k TM$ for tensor bundles. It is also always useful to recall that the tensor product of spaces can be identified with the space of multilinear, real-valued maps, so that

$$\otimes^k TM = TM \otimes \cdots \otimes TM \cong L(T^*M, \ldots, T^*M; \mathbb{R}).$$

5 Vector Fields and flows

(Lecture 4)

Recall that a vector field $X \in \Gamma(TM)$ is a differential operator $X : C^\infty(M) \to C^\infty(M)$. In local coordinates $(x^1, \ldots, x^n)$, a vector field can be expressed as $X = X^i \frac{\partial}{\partial x^i}$.

> **Definition 5.1** (Lie Bracket/Commutator of Vector Fields). Let $X, Y \in \Gamma(TM)$. The Lie bracket (or commutator) $[-,-] : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ of X and Y is defined by
>
> $$[X, Y] := X \circ Y - Y \circ X$$
>
> where X and Y are viewed as differential operators on $C^\infty(M)$.

Although $X \circ Y$ is a second order differential operator, $[X, Y]$ is a vector field (first order differential operator). Which can be seen by considering its coordinate expression.

$$\begin{aligned}[X,Y]^h &= X(Y^h) - Y(X^h) \\ &= X^i \frac{\partial}{\partial x^i}(Y^h \frac{\partial}{\partial x^h}) - Y^i \frac{\partial}{\partial x^i}(X^h \frac{\partial}{\partial x^h}) \\ &= X^i \frac{\partial Y^h}{\partial x^i} - Y^i \frac{\partial X^h}{\partial x^i}.\end{aligned}$$

Remark 5.2. Properties (see exercise 3.3):

- $\mathbb{R}$ linearity
- $[X, Y] = -[Y, X]$ (anti-commutativity/-symmetry)
- Jacobi's identity
$$[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0$$
- $[fX, gY] = f \cdot g[X, Y] + X(g) \cdot f \cdot Y - Y(f) \cdot g \cdot X$

If $[X, Y]$ were a tensor field, then it would have to be of type (1,2). But $[fX, Y] \neq f[X, Y]$, as it involves derivatives of f. So $[-,-]$ is not tensorial.

One can easily show that for X, Y being coordinate vector fields, the Lie bracket vanishes, i.e. in any local coordinate chart

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0. \tag{5.1}$$

Indeed, let (U, x) be a coordinate chart on a smooth manifold M. Recall that $\frac{\partial}{\partial x^i}(x^j) = \delta_i^j$ is constant. Hence, for any coordinate function x^k,

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right](x^k) = \frac{\partial}{\partial x^i}\left(\frac{\partial}{\partial x^j}(x^k)\right) - \frac{\partial}{\partial x^j}\left(\frac{\partial}{\partial x^i}(x^k)\right) = \frac{\partial}{\partial x^i}(\delta_j^k) - \frac{\partial}{\partial x^j}(\delta_i^k) = 0.$$

Thus the bracket vanishes on all coordinate functions and is therefore the zero field.

$(\Gamma(TM), [-,-])$ is an infinite-dimensional Lie algebra with the Lie group of diffeomorphisms of M to itself, Diff(M), provided that M is compact. (The Lie group is a manifold equipped with a compatible group structure such that the group operations are smooth maps.)

5.1 Flow of a vector field

Let $X \in \Gamma(TM)$.

Definition 5.3 (Integral Curve of a Vector Field). A curve $\gamma : (a,b) \longrightarrow M$ is an integral curve of X if

$$\dot{\gamma}(t) = X \circ \gamma(t), \quad \forall t \in (a,b).$$

In local coordinates:

$$(\dot{\gamma})^h(t) = X^h(\gamma(t)).$$

We recall the following result from classic ODE theory.

Theorem 5.4 (Local Well-Posedness of ODEs). Let $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be smooth. For every $u_0 \in \mathbb{R}^n$, the IVP

$$u'(t) = F(t, u(t)), \quad u(0) = u_0$$

admits a unique maximal solution $u : (T_-, T_+) \to \mathbb{R}^n$, where $T_\pm \in (0, \infty]$, such that:

- Maximality: u cannot be extended beyond (T_-, T_+).
- Blow-up alternative: If $T_+ < \infty$ (resp. $T_- > -\infty$), then $|u(t)| \to \infty$ as $t \uparrow T_+$ (resp. $t \downarrow T_-$).
- Smooth dependence: For any compact $[a,b] \subset (T_-, T_+)$, the map $u_0 \mapsto u|_{[a,b]}$ is smooth.

Characterisation of maximality: the blow-up alternative is equivalent to the statement that if

$$\sup_{t \in [0, T_+)} |u(t)| < \infty,$$

then $T_+ = +\infty$ (and similarly for T_-). In other words, a maximal solution can be extended past any finite time unless its norm becomes unbounded/as long as it remains bounded.
Moreover, $u(t) = u(t; u_0)$ depends smoothly on its arguments. This means that the *flow map* $\Phi(t, u_0) := u(t; u_0)$ is smooth in both t and u_0. However, the flow is not defined on a uniform time interval for all initial conditions u_0; rather, for each u_0, there exists some time interval $(-T_-(u_0), T_+(u_0))$ on which the flow is defined.

By working in local coordinates, the above theorem implies the following:

Theorem 5.5 (Flow of a Vector Field). Let $X \in \Gamma(TM)$. For every open, precompact subset $U \subset M$ (compact closure $\overline{U}$), there exist $\delta > 0$ and a unique smooth map

$$\Phi : (-\delta, \delta) \times \overline{U} \longrightarrow M$$

called the flow map, such that for all $q \in \overline{U}$, the curve $\gamma(t) = \Phi_t(q)$ is an integral curve of X starting from q, i.e.,

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\Phi_t(q) = X|_{\Phi_t(q)} \\ \Phi_0(q) = q \end{cases}$$

Moreover, if X is compactly supported, then $U = M$ and $\delta = +\infty$.

Proof. See [TR24], chapter 7.3. □

Remark 5.6. Two integral curves cannot intersect (uniqueness of solutions of ODEs) $\implies$ Φ_t is a diffeomorphism on its image.

Moreover, they form a semigroup, meaning that for all $t_1, t_2 \in (-\delta, \delta)$ and $q \in \overline{U}$, we have

$$\Phi_{t_1+t_2}(q) = (\Phi_{t_1} \circ \Phi_{t_2})(q).$$

Example 5.7. If $X = \frac{\partial}{\partial x^1}$, we solve the flow equation. The differential equation for the flow $\Phi_t(x) = (x^1(t), \ldots, x^n(t))$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t^k(x) = X^k(\Phi_t(x))$$

where X^k are the components of X. Since $X = \frac{\partial}{\partial x^1}$, we have $X^k = \delta_1^k$ and in particular $X^k(\Phi_t(x)) = \delta_k^1$, so:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t^k(x) = \delta_1^k = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k \neq 1 \end{cases}$$

With initial condition $\Phi_0(x^1, \ldots, x^n) = (x^1, \ldots, x^n)$, integrating gives:

$$\Phi_t((x^1, \ldots, x^n)) = (x^1 + t, x^2, \ldots, x^n).$$

So the flow translates points in the x^1-direction at unit speed, as expected.

Lemma 5.8 (Straightening a Vector Field). Let $X \in \Gamma(TM)$ with $X_p \neq 0$ at some point $p \in M$. Then there exists a local coordinate system around p in which $X = \frac{\partial}{\partial x^1}$.

Proof. See exercise 2.4. □

Lemma 5.9 (Commuting Vector Fields Generate Coordinates). Let $X, Y \in \Gamma(TM)$ be smooth, $[X, Y] = 0$, and let X_p and Y_p be linearly independent at $p \in M$, then there exists local coordinates (x^1, x^2) around p such that $X = \frac{\partial}{\partial x^1}$ and $Y = \frac{\partial}{\partial x^2}$.

Proof. See exercise 4.4. □

If we drop the compactness assumption, the flow from Theorem 5.5 doesn't necessarily exist for all times. To see this consider the following example.

Example 5.10. On $\mathbb{R}$: Let $X = y^2 \frac{\partial}{\partial y}$ and $y_0 > 0$. The integral curve $y(t)$ satisfies $\dot{y} = y^2$, which gives

$$y(t) = \frac{1}{\frac{1}{y_0} - t}.$$

This solution is only defined for $t < \frac{1}{y_0}$, showing that the flow reaches infinity in finite time.

A vector field generates the flow(s), a one-parameter family of diffeomorphisms. One can form the commutator on the group of such diffeomorphisms.

Recall now Definition 5.1.

Remark 5.11. One can show that in local coordinates, the commutator of two flows can be expressed as

$$[X, Y]^k = \lim_{t,s \to 0} \frac{\left(\Phi_{-s}^{(Y)} \circ \Phi_{-t}^{(X)} \circ \Phi_s^{(Y)} \circ \Phi_t^{(X)}(p)\right)^k - p^k}{ts}.$$

This expression quantifies how much the flows of X and Y fail to commute. The four successive flows applied in sequence would return to p if $[X, Y] = 0$; otherwise, we drift by an amount proportional to ts. The Lie bracket captures the leading-order term in this drift. Thus the geometric notion of flow non-commutativity coincides with the algebraic definition of the commutator.

5.2 Lie derivative

Definition 5.12 (Lie Derivative). Let $X \in \Gamma(TM)$ and $\{\phi_t\}_{t\in\mathbb{R}}$ be its flow. The Lie derivative is defined by:

- For $f \in C^\infty(M)$:
$$\mathcal{L}_X f(p) = \lim_{t\to 0} \frac{f(\phi_t(p)) - f(p)}{t}$$
- For $Y \in \Gamma(TM)$:
$$(\mathcal{L}_X Y)_p = \lim_{t\to 0} \frac{(\mathrm{d}\phi_{-t})_{\phi_t(p)}(Y|_{\phi_t(p)}) - Y|_p}{t} = \lim_{t\to 0} \frac{Y|_{\phi_t(p)} - \mathrm{d}(\phi_{+t})_p(Y|_p)}{t}$$

Theorem 5.13 (Lie Derivative of Functions and Vector Fields). Let $X \in \Gamma(TM)$ with flow $\{\phi_t\}_{t\in\mathbb{R}}$. Then:

$$\mathcal{L}_X f = X(f) \quad \text{for all } f \in C^\infty(M),$$
$$\mathcal{L}_X Y = [X, Y] \quad \text{for all } Y \in \Gamma(TM).$$

Proof. See handwritten notes from the lecture. □

Lemma 5.14 (Extension of Lie Derivative to Tensor Fields). The Lie derivative $\mathcal{L}_X$ extends to all tensor fields by requiring two *axioms*:

1. For $\omega \in \Gamma(T^*M)$: $(\mathcal{L}_X\omega)(Y) := \mathcal{L}_X(\omega(Y)) - \omega(\mathcal{L}_X Y)$
2. Leibniz rule: $\mathcal{L}_X(T_1 \otimes T_2) = (\mathcal{L}_X T_1) \otimes T_2 + T_1 \otimes (\mathcal{L}_X T_2)$

This yields a map
$$\mathcal{L} : \Gamma(TM) \times T^k_l(M) \longrightarrow T^k_l(M)$$
with $\mathcal{L}_X$ commuting with contractions.

Note that this extension is unique, but we will not prove this here.
We remark that the first axiom is natural, because $\mathcal{L}_X(\omega(Y)) = (\mathcal{L}_X\omega)(Y) + \omega(\mathcal{L}_X Y)$ mimics a sort of chain rule.

Proof sketch. Build inductively using the two axioms. For (k, l)-tensors, decompose as products of vectors and covectors, apply the Leibniz rule. □

Further more, this construction respects contractions (Definition 4.19). Indeed, if $w \in \Gamma(\mathrm{T}^*M)$, $Y \in \Gamma(\mathrm{T}M)$, then $w \otimes Y$ is of type $(1, 1)$, and in coordinates we have

$$(w \otimes Y)^\alpha_\beta = w_\beta Y^\alpha.$$

If one applies the contraction one gets

$$\mathrm{tr}(w \otimes Y) = (w \otimes Y)^\alpha_\alpha = w_\alpha Y^\alpha.$$

But

$$w(Y) = w_\beta dx^\beta \left(Y^\alpha \frac{\partial}{\partial x^\alpha} \right) = w_\alpha Y^\alpha.$$

And therefore

$$\operatorname{tr}(w \otimes Y) = w(Y). \tag{5.2}$$

Now apply $\mathcal{L}_X$ to both sides and apply the first axiom/requirement,

$$\begin{aligned} \mathcal{L}_X(\operatorname{tr}(w \otimes Y)) &= \mathcal{L}_X(w(Y)) \\ &\overset{1.}{=} (\mathcal{L}_X w)(Y) + w(\mathcal{L}_X Y) \end{aligned}$$

Now we use the second axiom/requirement and apply the contraction on both sides.

$$\begin{aligned} \operatorname{tr}(\mathcal{L}_X(w \otimes Y)) &\overset{2.}{=} \operatorname{tr}((\mathcal{L}_X w) \otimes Y) + \operatorname{tr}(w \otimes (\mathcal{L}_X Y)) \\ &\overset{(5.2)}{=} (\mathcal{L}_X w)(Y) + w(\mathcal{L}_X Y) \end{aligned}$$

That way, comparing the two expressions, we have established the following lemma.

Lemma 5.15 (Lie Derivative Commutes with Contraction for $(1,1)$ Tensors). Let $w \in \Gamma(\mathrm{T}^*M)$ and $Y \in \Gamma(\mathrm{T}M)$. Then the Lie derivative commutes with contraction:

$$\mathcal{L}_X(\operatorname{tr}(w \otimes Y)) = \operatorname{tr}(\mathcal{L}_X(w \otimes Y)).$$

One can also construct this extension the other way round, i.e. we require the Leibnitz rule + the contraction compatibility, and we get the first axiom as a consequence.

$$\begin{aligned} \mathcal{L}_X w(Y) &\overset{(5.2)}{=} \mathcal{L}_X(\operatorname{tr}(w \otimes Y)) \\ &= \operatorname{tr}(\mathcal{L}_X(w \otimes Y)) \\ &= \operatorname{tr}(\mathcal{L}_X w \otimes Y) + \operatorname{tr}(w \otimes \mathcal{L}_X Y) \\ &= (\mathcal{L}_X w)(Y) + w(\mathcal{L}_X Y) \end{aligned}$$

Remark 5.16. The results from before can be summarised as follows. We have 4 statements.

1. $(\mathcal{L}_X \omega)(Y) := \mathcal{L}_X(\omega(Y)) - \omega(\mathcal{L}_X Y)$
2. Leibnitz rule for tensor products
3. $\mathcal{L}_X$ commutes with contractions
4. Extension of $\mathcal{L}_X$ to all tensor fields

Then we have

$$1.\text{ and }2. \iff 3.\text{ and }4. \iff (2.\text{ and }3. \implies 1.)$$

Remark 5.17. In general: Tensors evaluated at their arguments, like $w(Y)$, $g(X,Y)$, can be viewed as contractions of tensor products. E.g.:

$$g(X,Y) = g_{ab} X^a Y^b = \operatorname{tr}(\operatorname{tr}(g \otimes X \otimes Y))$$

Remark 5.18. If $T \in T^k_l(M)$, and $X = \frac{\partial}{\partial x^i}$, in local coordinates then

$$(\mathcal{L}_X T)^{i_1 \dots i_k}_{j_1 \dots j_l} = \frac{\partial T^{i_1 \dots i_k}_{j_1 \dots j_l}}{\partial x^i}.$$

See the solution of exercise 4.3b.

In fact we never need a general coordinate formula. Locally, any tensor is a $C^\infty(M)$-linear combination of tensor products of the coordinate fields ∂_i and $\mathrm{d}x^j$, so the Leibniz rule reduces *every* computation of a Lie derivative to the action of $\mathcal{L}_X$ on these three building blocks.
For $X = X^m\partial_m$, the Lie derivative $\mathcal{L}_X$ is determined by $\mathbb{R}$-linearity and the Leibniz rule together with its action on functions and on the coordinate basis fields:

$$\mathcal{L}_X f = X^m\,\partial_m f, \qquad \mathcal{L}_X\partial_i = -(\partial_i X^m)\,\partial_m, \qquad \mathcal{L}_X \mathrm{d}x^j = (\partial_i X^j)\,\mathrm{d}x^i. \tag{5.3}$$

The first two identities follow from Theorem 5.13; the third follows from the covector axiom of Lemma 5.14: applying the LHS to a vector field ∂_i gives $(\mathcal{L}_X \mathrm{d}x^j)(\partial_i) = X(\delta^j_i) - \mathrm{d}x^j([X,\partial_i]) = \ldots = \partial_i X^j$, and the latter is exactly what we get if we evaluate the RHS on ∂_i.
To differentiate an arbitrary tensor, expand it in the coordinate basis and apply the Leibniz rule to each factor.

Example 5.19 (Lie derivative of the metric). For instance, for the metric $g = g_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j$,

$$\begin{aligned}\mathcal{L}_X g &= (\mathcal{L}_X g_{ij})\,\mathrm{d}x^i \otimes \mathrm{d}x^j + g_{ij}\,(\mathcal{L}_X \mathrm{d}x^i) \otimes \mathrm{d}x^j + g_{ij}\,\mathrm{d}x^i \otimes (\mathcal{L}_X \mathrm{d}x^j)\\ &= \big(X^m\,\partial_m g_{ij} + g_{mj}\,\partial_i X^m + g_{im}\,\partial_j X^m\big)\,\mathrm{d}x^i \otimes \mathrm{d}x^j,\end{aligned}$$

where in step 1 we used the Leibniz rule and in step 2 we used the first and third identity in equation (5.3).

In particular, X is a Killing field (Definition 5.21) precisely when the bracketed expression vanishes.

Lemma 5.20 (Lie Derivative Commutator). For any $X, Y \in \Gamma(TM)$ and tensor field $T \in T^k_l(M)$, the Lie derivatives with respect to X and Y satisfy

$$\mathcal{L}_{[X,Y]}T = \mathcal{L}_X\mathcal{L}_Y T - \mathcal{L}_Y\mathcal{L}_X T.$$

Proof. See exercise 4.3a. □

Definition 5.21 (Killing Vector Field). Let (M, g) be a Riemannian manifold. A vector field $X \in \Gamma(\mathrm{T}M)$ is called a *Killing vector field* if

$$\mathcal{L}_X g = 0.$$

Remark: this is named after Wilhelm Killing, and not after literally *killing* a vector field.

Lemma 5.22 (Isometric flows and Killing Vector Fields). Let $X \in \Gamma(\mathrm{T}M)$ with flow $\{\Phi_t\}$. If X generates a flow of isometries, it is Killing, i.e.

$$(\Phi_t)^*(g|_{\Phi_t}) = g \implies \mathcal{L}_X g = 0.$$

Proof. See exercise 4.3b. □

We will not show the converse in full generality, but we proved a related statement in exercise 5.1. There we show that if a metric is independent of a coordinate x^k, then the flow F_t of $\frac{\partial}{\partial x^k}$ is an isometry, $F_t^*(g|_{F_t}) = g$. We use this several times, for example in exercise 7.2b.

5.3 Connections

For vector fields V and Y, $\mathcal{L}_V Y$ is not suitable as a directional derivative because it is not $C^\infty(M)$-linear in the first argument: for $f \in C^\infty(M)$,

$$\mathcal{L}_{fV} Y \neq f \mathcal{L}_V Y,$$

so $\mathcal{L}_V Y|_p$ depends on V near p, not just on $V|_p$. Given $Y \in \Gamma(TM)$, $p \in M$, and $V \in T_p M$, we want a directional derivative $D_V Y|_p$ that depends only on $V|_p$. This requires comparing tangent vectors at different points, which is not possible immediately, as they live in different spaces. A systematic way to do this defines a *connection*, enabling differentiation of vector fields, curve acceleration, and parallel transport.

Definition 5.23 (Connection). A connection ∇ on M is a map

$$\nabla : \Gamma(\mathrm{T}M) \times \Gamma(\mathrm{T}M) \to \Gamma(\mathrm{T}M)$$

such that:

1. ∇ is $C^\infty(M)$-linear in the first argument:
$$\nabla_{X_1 + f \cdot X_2} Y = \nabla_{X_1} Y + f \cdot \nabla_{X_2} Y$$
2. ∇ is $\mathbb{R}$-linear in the second argument:
$$\nabla_X (Y_1 + aY_2) = \nabla_X Y_1 + a \nabla_X Y_2, \quad a \in \mathbb{R}$$
3. Satisfies the Leibniz rule with respect to the second argument:
$$\nabla_X (fY) = X(f) \cdot Y + f \cdot \nabla_X Y$$

It is also called *covariant derivative*.

∇Y is a $(1,1)$-tensor field. It is also local: if $U \subseteq M$ is open and $p \in U$, then $(\nabla_X Y)|_p$ depends only on $X|_p$ and $Y|_U$.

For the existence of connections, see Theorem 5.34.
Because of the Leibniz rule in the second argument, connections are not $C^\infty(M)$-linear in the second argument (there is this extra term $X(f) \cdot Y$) and therefore they are not tensor fields.
On $\mathbb{R}^n$ one can verify that the *Euclidean connection* defined via the standard directional derivative D

$$(D_X Y)^k = X^i \frac{\partial Y^k}{\partial x^i}. \tag{5.4}$$

is in fact a connection.

Lemma 5.24. If ∇ is a connection on M, it defines a unique connection on any open subset $U \subseteq M$.

Proof. Let $X, Y \in \Gamma(\mathrm{T}U)$, $p \in U$. We want to show that $\nabla_X Y|_p$ is uniquely defined.
Let $F : M \to [0,1]$ be a smooth cutoff function such that

$$F(q) = \begin{cases} 1 \text{ if } q \text{ is in a neighborhood of } p \text{ in } U \\ 0 \text{ on } M \setminus U. \end{cases}$$

Then $\widetilde{X} = F \cdot X$, $\widetilde{Y} = F \cdot Y$ extend to the whole of M.

Define $\nabla_X Y|_p = \nabla_{\widetilde{X}} \widetilde{Y}|_p$. This is independent of the choice of F: If $Y, \widetilde{Y}$ are two vector fields such that $\widetilde{Y} = Y$ in a neighbourhood V of p and if $f : M \to [0,1]$,

$$f = \begin{cases} 1 & \text{on } V' \subset V \\ 0 & \text{on } M \setminus V \end{cases}$$

then

$$\nabla_X \left(f \cdot (Y - \widetilde{Y}) \right)\Big|_p = X(f)|_p \cdot \underbrace{(Y - \widetilde{Y})|_p}_{=0} + \underbrace{f|_p}_{=1} \cdot (\nabla_X (Y - \widetilde{Y})\Big|_p)$$

But $f \cdot (Y - \widetilde{Y}) = 0$ in a neighborhood of $p \implies \nabla_X Y|_p = \nabla_X \widetilde{Y}\Big|_p$

□

We will use the above in order to talk about $\nabla_X Y$ when X, Y are not defined on the whole of M (e.g. coordinate vector fields), or fields on a chart.

Definition 5.25 (Christoffel Symbols). Let $(x^1, \ldots, x^n)$ be a local coordinate system. The Christoffel symbols Γ^k_{ij} are defined by

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k}.$$

Equivalently,

$$\Gamma^k_{ij} = \mathrm{d}x^k \left(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \right) = \left(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \right)^k$$

is the k^{th} component.
Note: Γ^k_{ij} are not components of a tensor (see exercise 4.1a). If they would, having zero Christoffel symbols in one coordinate system (say cartesian) would imply they vanish in all coordinate systems, which is not the case.

Using the Leibnitz rule from Definition 5.23, we can easily see that the covariant derivative of a vector field Y in the direction of X is given by

$$(\nabla_X Y)^k = X^i \frac{\partial Y^k}{\partial x^i} + \Gamma^k_{ij} X^i Y^j. \tag{5.5}$$

We will now investigate how the connection acts on general tensor fields and provide coordinate formulas.

Lemma 5.26 (Extension of Connections to Tensor Fields). Let M be a smooth manifold equipped with a connection ∇. The connection extends to a map

$$\nabla : \Gamma(TM) \times T^k_l(M) \longrightarrow T^k_l(M)$$

by requiring:

1. The Leibniz rule for tensor products: $\nabla_X (S \otimes T) = (\nabla_X S) \otimes T + S \otimes (\nabla_X T)$
2. Compatibility with contractions: $\nabla_X (\operatorname{tr} A) = \operatorname{tr}(\nabla_X A)$

As we show in exercise 4.2, the coordinate formula for the connection of a general tensor T in the direction $X = \frac{\partial}{\partial x^a}$ is

$$(\nabla_{\partial_a} T)^{i_1 \ldots i_k}_{j_1 \ldots j_l} = \partial_a T^{i_1 \ldots i_k}_{j_1 \ldots j_l} + \sum_{r=1}^{k} \Gamma^{i_r}_{ab} T^{i_1 \ldots b \ldots i_k}_{j_1 \ldots j_l} - \sum_{s=1}^{l} \Gamma^b_{a j_s} T^{i_1 \ldots i_k}_{j_1 \ldots b \ldots j_l}. \tag{5.6}$$

Proof of the Lemma. See Exercise 4.2. □

For a function $f \in C^\infty(M)$, viewed as a $(0,0)$-tensor, the connection acts as the directional derivative, $\nabla_X f = X(f)$. This is forced by comparing two expansions of $\nabla_X(fY)$: the Leibniz rule of Definition 5.23 gives $\nabla_X(fY) = X(f)\,Y + f\,\nabla_X Y$, while viewing $fY = f \otimes Y$ and applying the tensor Leibniz rule from Lemma 5.26 gives $\nabla_X(fY) = (\nabla_X f)\,Y + f\,\nabla_X Y$. The common term $f\,\nabla_X Y$ cancels, leaving $(\nabla_X f)\,Y = X(f)\,Y$ for all Y, hence $\nabla_X f = X(f)$. Writing $X = X^i \frac{\partial}{\partial x^i}$, the coordinate expression is therefore

$$\nabla_X f = X(f) = X^i \frac{\partial f}{\partial x^i}. \tag{5.7}$$

Similarly, the action of ∇ on a basis 1-form $\mathrm{d}x^j$ is determined once we know it on functions and on the basis vector fields. Since $\mathrm{d}x^j(\partial_i) = \delta^j_i$ is a constant function (which vanishes under the connection, see (5.7)) and the pairing $\omega(Y)$ is a contraction of $\omega \otimes Y$, the contraction compatibility of Lemma 5.26 gives

$$\begin{aligned}
0 &= \nabla_{\partial_a}(\delta^j_i) \\
&= \nabla_{\partial_a}\big(\mathrm{d}x^j(\partial_i)\big) \\
&= \nabla_{\partial_a}\big(\operatorname{tr}(\mathrm{d}x^j \otimes \partial_i)\big) \\
&= \operatorname{tr}\big(\nabla_{\partial_a}(\mathrm{d}x^j \otimes \partial_i)\big) \\
&= \operatorname{tr}\big((\nabla_{\partial_a}\mathrm{d}x^j) \otimes \partial_i + \mathrm{d}x^j \otimes (\nabla_{\partial_a}\partial_i)\big) \\
&= \operatorname{tr}\big((\nabla_{\partial_a}\mathrm{d}x^j) \otimes \partial_i\big) + \operatorname{tr}\big(\mathrm{d}x^j \otimes (\nabla_{\partial_a}\partial_i)\big) \\
&= (\nabla_{\partial_a}\mathrm{d}x^j)(\partial_i) + \mathrm{d}x^j(\nabla_{\partial_a}\partial_i) \\
&= (\nabla_{\partial_a}\mathrm{d}x^j)(\partial_i) + \Gamma^j_{ai}.
\end{aligned}$$

In the very last step we have used Definition 5.25 to find $\nabla_{\partial_a}\partial_i = \Gamma^k_{ai}\,\partial_k$, and then $\mathrm{d}x^j(\Gamma^k_{ai}\,\partial_k) = \Gamma^k_{ai}\,\mathrm{d}x^j(\partial_k) = \Gamma^k_{ai}\,\delta^j_k = \Gamma^j_{ai}$. So we have shown

$$(\nabla_{\partial_a}\mathrm{d}x^j)(\partial_i) = -\Gamma^j_{ai}.$$

As a 1-form is determined by its values on the basis fields, this means

$$\nabla_{\partial_a}\mathrm{d}x^j = -\Gamma^j_{ak}\,\mathrm{d}x^k. \tag{5.8}$$

To compute the connection of a general tensor wrt. a general vector field X in practice, i.e. in local coordinates, we have two options:

1. Unlike the Lie derivative, ∇ is $C^\infty(M)$-linear in its direction argument, $\nabla_{fX} = f\,\nabla_X$, so $\nabla_X T = X^a\,\nabla_{\partial_a} T$ and we can use (5.6).
2. Alternatively, to avoid having to memorise formula (5.6), we can proceed similarly as for the Lie derivative (see (5.3) and Example 5.19) and use the Leibniz rule to reduce to the coordinate basis and then apply identites (5.7), (5.5), and (5.8), namely

$$\nabla_{\partial_a} f = \partial_a f, \qquad \nabla_{\partial_a}\partial_i = \Gamma^k_{ai}\,\partial_k, \qquad \nabla_{\partial_a}\mathrm{d}x^j = -\Gamma^j_{ak}\,\mathrm{d}x^k, \tag{5.9}$$

   together with the $C^\infty(M)$-linearity $\nabla_X T = X^a\,\nabla_{\partial_a} T$.

On a manifold, there are possibly infinitely many connections. So, this recipe is as far as we can go with a general connection: the Christoffel symbols are *not* determined by the connection axioms, but are precisely the data specifying the connection (different connections have different Γ^k_{ij}), so the recipe reduces every computation to them without computing them. For the canonical connection of a Riemannian manifold, the Levi-Civita connection, the Christoffel symbols are instead given explicitly in terms of the metric; see Theorem 5.34 and formula (5.13) in the next lecture.

Example 5.27 (Covariant derivative of the metric). Using option 2 and the identities (5.9), we compute the covariant derivative of a general metric $g = g_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j$ in the direction of a general vector field $X = X^a \partial_a$, exactly as we did for the Lie derivative in Example 5.19. The Leibniz rule gives

$$\begin{aligned}
\nabla_X g &= (\nabla_X g_{ij})\, \mathrm{d}x^i \otimes \mathrm{d}x^j + g_{ij}\,(\nabla_X \mathrm{d}x^i) \otimes \mathrm{d}x^j + g_{ij}\, \mathrm{d}x^i \otimes (\nabla_X \mathrm{d}x^j) \\
&= X^a(\nabla_{\partial_a} g_{ij})\, \mathrm{d}x^i \otimes \mathrm{d}x^j + g_{ij}\, X^a(\nabla_{\partial_a} \mathrm{d}x^i) \otimes \mathrm{d}x^j + g_{ij}\, X^a\, \mathrm{d}x^i \otimes (\nabla_{\partial_a} \mathrm{d}x^j) \\
&= X^a(\partial_a g_{ij})\, \mathrm{d}x^i \otimes \mathrm{d}x^j - g_{ij}\, X^a \Gamma^i_{ak}\, \mathrm{d}x^k \otimes \mathrm{d}x^j - g_{ij}\, X^a \Gamma^j_{ak}\, \mathrm{d}x^i \otimes \mathrm{d}x^k \\
&= X^a\big(\partial_a g_{ij} - \Gamma^m_{ai}\, g_{mj} - \Gamma^m_{aj}\, g_{im}\big)\, \mathrm{d}x^i \otimes \mathrm{d}x^j,
\end{aligned}$$

where in the second step we used the $C^\infty(M)$-linearity $\nabla_X = X^a\, \nabla_{\partial_a}$, in the third the building blocks (5.7) and (5.8), and in the fourth we relabelled the summation indices. And of course the components are $(\nabla_X g)_{ij} = X^a\big(\partial_a g_{ij} - \Gamma^m_{ai}\, g_{mj} - \Gamma^m_{aj}\, g_{im}\big)$.

Remark 5.28 (Lie derivative vs. covariant derivative). Both $\mathcal{L}_X$ and ∇_X turn a direction field X and a tensor into a tensor of the same type, but they differ fundamentally. Comparing the two computations for the metric (Examples 5.19 and 5.27),

$$\begin{aligned}
(\mathcal{L}_X g)_{ij} &= X^m\, \partial_m g_{ij} + g_{mj}\, \partial_i X^m + g_{im}\, \partial_j X^m, \\
(\nabla_X g)_{ij} &= X^a\big(\partial_a g_{ij} - \Gamma^m_{ai}\, g_{mj} - \Gamma^m_{aj}\, g_{im}\big),
\end{aligned}$$

Some differences stand out:

1. *Dependence on X.* In $\nabla_X g$ the field X enters only through the pointwise factor X^a, so $(\nabla_X g)|_p$ depends only on $X|_p$ (indeed ∇ is $C^\infty(M)$-linear in X). In $\mathcal{L}_X g$ the derivatives $\partial_i X^m$ appear, so $(\mathcal{L}_X g)|_p$ depends on X in a whole neighbourhood of p ($\mathcal{L}$ is only $\mathbb{R}$-linear in X). Only ∇_X is a genuine generalisation of the directional derivative.
2. *Required data.* $\mathcal{L}_X$ uses only the smooth structure (via the flow of X), whereas ∇_X requires a choice of connection, i.e. the Christoffel symbols Γ^k_{ij}.
3. Further, on vector fields the two are related directly: $\mathcal{L}_X Y = [X, Y] = \nabla_X Y - \nabla_Y X$ whenever ∇ is *torsion-free*, see Definition 5.29 below.

Definition 5.29 (Torsion of a Connection). Let ∇ be a connection on M. The torsion tensor T is defined by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

for all $X, Y \in \Gamma(TM)$. Here the term $\nabla_Y X + [X, Y]$ is called *adjoint torsion*.

T is an anti-symmetric $(1, 2)$ tensor, see exercise 4.1b.

Remark 5.30 (Torsion for coordinate vector fields). For coordinate vector-fields,

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0$$

so in view of Definition 5.25, we have

$$T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji},$$

and there fore also

$$T = 0 \iff \Gamma^k_{ij} = \Gamma^k_{ji}.$$

Suffices to be true in one coordinate system.

(Lecture 5)

Definition 5.31 (Vector Field Along a Curve). Let $\gamma : I \to M$ be a smooth curve. We call a smooth map

$$X : t \longmapsto X(t) \in T_{\gamma(t)}M$$

a *vector field along a curve* for $t \in I$, assigning to each t a tangent vector at $\gamma(t)$.

A vector field along a curve can equivalently be seen as a section $t \mapsto (t, X(t))$ of the bundle γ^*TM, the latter being the pullback via γ of the tangent bundle

$$\gamma^*TM = \{(t, v) \in I \times TM : v \in T_{\gamma(t)}M\}.$$

Each fiber $\pi^{-1}(t) = t \times T_{\gamma(t)}M \cong T_{\gamma(t)}M$ of γ^*TM is a vector space, where the bundle projection is given by $\pi : \gamma^*TM \to I$, $(t, v) \mapsto t$.
Hence we can denote the space of all vector fields along γ by $\Gamma(\gamma^*TM)$.

Remark 5.32. If γ self-intersects (i.e., $\gamma(t_1) = \gamma(t_2)$), $X(t_1)$ does not have to be the same as $X(t_2)$ (so not a unique value at all points of M belonging to the image of γ).
In this case: X does not extend to a vector field on M. If γ is an embedded curve, X always extends (non-uniquely, i.e. in many different ways) to an element of $\Gamma(TM)$.
Locally (i.e., restricting the domain of γ): Any curve is embedded.

Lemma 5.33 (Covariant Derivative Along a Curve). Let $\gamma : I \to M$ be a smooth curve and X a vector field along γ. The covariant derivative $\nabla_{\dot\gamma}X$ is uniquely defined at each $t \in I$, independently of any local extension of X around $\gamma(t)$.

Sketch of proof. Assuming $\dot\gamma(t) \neq 0$, restrict to $\gamma|_{(t-\epsilon,t+\epsilon)}$ so that it is embedded curve.
Extend X to a vector field in a neighborhood of $\gamma(t)$ in M. Then, in local coordinates around $\gamma(t)$:

$$\begin{aligned}(\nabla_{\dot\gamma}X|_{\gamma(t)})^k &= \dot\gamma^i \frac{\partial}{\partial x^i} X^k + \Gamma^k_{ij}(\gamma(t))\, \dot\gamma^i X^j \\ &= \frac{\mathrm{d}}{\mathrm{d}t} X^i_t + \Gamma^k_{ij}(\gamma(t))\, \dot\gamma^i X^j_t\end{aligned}$$

The last expression depends only on the values along γ. □

Levi-Civita connection

On a general manifold: An infinite family of connections. The difference of two such connections, $\nabla - \overline{\nabla}$ is always a (1,2)-tensor field (see exercise 4.1c). If M is equipped with a Riemannian metric, there exists a unique *natural* connection.

Theorem 5.34 (Levi-Civita Connection). Let (M, g) be a Riemannian manifold. There exists a *unique* connection ∇ (called the Levi-Civita connection) such that:

1. ∇ is torsion-free: $\Gamma^k_{ij} = \Gamma^k_{ji}$ (if true in one coordinate system, it's true for all; since the torsion is a tensor).

2. ∇ respects the metric: for any $X, Y, Z \in \Gamma(TM)$,

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

(equivalently, $\nabla_X g = 0$ for all X.)

Proof. Uniqueness: It suffices to prove that in any local coordinate system, conditions 1. & 2. uniquely fix the Christoffel symbols.
In any local coordinates $(x^1, \dots, x^n)$ we have

$$\begin{aligned}\frac{\partial}{\partial x^i} g_{jh} &= \frac{\partial}{\partial x^i}\left(g\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^h}\right)\right) \\ &= g\left(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^h}\right) + g\left(\frac{\partial}{\partial x^j}, \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^h}\right) \\ &= g\left(\Gamma^a_{ij}\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^h}\right) + g\left(\frac{\partial}{\partial x^j}, \Gamma^a_{ih}\frac{\partial}{\partial x^a}\right) \\ &= \Gamma^a_{ij} g_{ah} + \Gamma^a_{ih} g_{aj} \end{aligned} \tag{5.10}$$

Permuting indices:

$$\frac{\partial}{\partial x^h} g_{ij} = \Gamma^a_{hj} g_{ai} + \Gamma^a_{hi} g_{aj} \tag{5.11}$$

$$\frac{\partial}{\partial x^j} g_{ih} = \Gamma^a_{ji} g_{ah} + \Gamma^a_{jh} g_{ai} \tag{5.12}$$

So (5.11) + (5.12) - (5.10):

$$\frac{\partial}{\partial x^i} g_{jh} + \frac{\partial}{\partial x^j} g_{ih} - \frac{\partial}{\partial x^h} g_{ij} = 2\Gamma^a_{ij} g_{ah}$$

So i can solve this for Γ, as the above is just a matrix product. We obtain, after renaming the indices,

$$\Gamma^k_{ij} = \frac{1}{2} g^{kd}\left(\frac{\partial}{\partial x^i} g_{jd} + \frac{\partial}{\partial x^j} g_{id} - \frac{\partial}{\partial x^d} g_{ij}\right) \tag{5.13}$$

Where, as usual, $[g^{ij}]$ is the inverse matrix of $[g_{ij}]$. We see, that Γ is uniquely determined by the metric g.

Existence: Instead of using the above formula to construct a connection, we go a slightly different path. Let us use the shorthand notation $\langle \cdot, \cdot \rangle := g(\cdot, \cdot)$. In order to define $\nabla_X Y$, $X, Y \in \Gamma(TM)$, it suffices to define $\langle \nabla_X Y, Z \rangle$, $\forall X, Y, Z \in \Gamma(TM)$.

$$X\langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle \tag{5.14}$$

$$Y\langle X, Z \rangle = \langle \nabla_Y X, Z \rangle + \langle X, \nabla_Y Z \rangle \tag{5.15}$$

$$Z\langle X, Y \rangle = \langle \nabla_Z X, Y \rangle + \langle X, \nabla_Z Y \rangle \tag{5.16}$$

We use now the torsion-free condition, $\nabla_X Y - \nabla_Y X = [X, Y]$. From (5.14), (5.15), (5.16), we obtain Koszul's formula:

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2}\Big(X(\langle Y, Z \rangle) + Y(\langle X, Z \rangle) - Z(\langle X, Y \rangle) + \langle [X, Y], Z \rangle + \langle [Z, X], Y \rangle - \langle [Y, Z], X \rangle\Big)$$

One can use this formula to define a connection ∇ on M. It is essentially a coordinate-free definition of, the Levi-Civita connection, while equation (5.13) is its coordinate expression.

□

Example 5.35 (Euclidean connection on $\mathbb{R}^n$ is Levi-Civita). On $(\mathbb{R}^n, g_E)$, the standard directional derivative from equation (5.4), $(D_X Y)^k = X^i \frac{\partial Y^k}{\partial x^i}$, is the Levi-Civita connection. The Christoffel symbols are zero in this case, as one can also verify using (5.13).

Remark 5.36 (Levi-Civita Connection). The Levi-Civita connection is the canonical connection on (M, g). From now on, when referring to the connection of a Riemannian manifold, we will always mean the Levi-Civita connection.

Any construction based on a metric has to be preserved by isometries, as the latter identify geometric structures. In particular this holds for the Levi-Civita connection.

Lemma 5.37 (Isometries Preserve the Levi-Civita Connection). If $F : (M, g) \longrightarrow (N, h)$ is a (local) isometry and $X, Y \in \Gamma(TM)$, then

$$dF(\nabla_X^{(g)} Y) = \nabla_{dF(X)}^{(h)} dF(Y).$$

In short: an isometry leaves every term of Koszul's formula invariant, so it must map the Levi-Civita connection of g to that of h.

Proof sketch. Every term in Koszul's formula is respected by dF. Since F is a (local) isometry it preserves the inner product,

$$\langle dF(X), dF(Y)\rangle_h = \langle X, Y\rangle_g,$$

and, being a local diffeomorphism, its pushforward preserves the Lie bracket,

$$[dF(X), dF(Y)] = dF([X, Y]).$$

Hence Koszul's formula for $\nabla^{(h)}$, evaluated on $dF(X), dF(Y), dF(Z)$, reduces term by term to Koszul's formula for $\nabla^{(g)}$ on X, Y, Z, so that

$$\langle \nabla_{dF(X)}^{(h)} dF(Y),\ dF(Z)\rangle_h = \langle dF(\nabla_X^{(g)} Y),\ dF(Z)\rangle_h \qquad \text{for all } Z.$$

Since dF is a linear isomorphism on each tangent space, $dF(Z)$ ranges over all tangent vectors; as h is nondegenerate, the two sides agree as vectors,

$$dF(\nabla_X^{(g)} Y) = \nabla_{dF(X)}^{(h)} dF(Y).$$ □

Lemma 5.38 (Christoffel symbols via orthogonal projection for an immersion). Let $\Psi : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ be a smooth immersion with induced metric $g = \Psi^* g_E$, and let $\Pi^\top$ denote orthogonal projection onto $T\Psi(\Omega)$. Then

$$\Pi^\top\left(\frac{\partial}{\partial x^i}\frac{\partial \Psi}{\partial x^j}\right) = \Gamma_{ij}^k \frac{\partial \Psi}{\partial x^k}.$$

Proof. See exercise 5.4. □

5.4 Parallel transport

On a general manifold, parallel transport is the closest thing to a constant vector field along a curve.

Definition 5.39 (Parallel Transport). Let $\gamma : I \to M$ be smooth and ∇ any connection. For $s, t \in I$, the *parallel transport* along γ is the map

$$P_{s,t} : T_{\gamma(s)}M \longrightarrow T_{\gamma(t)}M, \qquad P_{s,t}(v) := X(t),$$

where X is the unique *parallel vector field along* γ, namely $\nabla_{\dot\gamma} X = 0$, and satisfies $X(s) = v$.

With this terminology, $X(t) = P_{s,t}(X(s))$ for every parallel field X along γ.

Theorem 5.40 (Existence and Uniqueness of Parallel Transport). For every $p \in M$, $v \in T_pM$, and smooth curve $\gamma : I \to M$ with $\gamma(0) = p$, there exists a unique vector field X along γ such that $X(0) = v$ and $\nabla_{\dot\gamma} X = 0$.

Proof. It suffices to do the construction in a local coordinate chart on M

$$\frac{\mathrm{d}}{\mathrm{d}t} X^k + \Gamma^k_{ij} \dot\gamma^i X^j = 0$$

This is a linear ODE, so ODE theory yields a unique solution. □

We note that the parallel transport has several nice properties, in particular for the Levi-Civita connection.

Lemma 5.41 (Basic Properties of Parallel Transport). Let $\gamma : I \to M$ be smooth, and let $P_{s,t} : T_{\gamma(s)}M \longrightarrow T_{\gamma(t)}M$ denote parallel transport along γ.

1. For all $s, t \in I$, $P_{s,t}$ is a linear isomorphism, with inverse $P_{t,s}$.
2. If ∇ is the Levi-Civita connection of (M, g), then $P_{s,t}$ is an isometry:

$$g_{\gamma(t)}\big(P_{s,t}v, P_{s,t}w\big) = g_{\gamma(s)}(v, w), \qquad v, w \in T_{\gamma(s)}M.$$

Proof. Parallel transport along γ is defined by a linear ODE in each coordinate chart,

$$\frac{\mathrm{d}}{\mathrm{d}t} X^k + \Gamma^k_{ij}(\gamma(t))\, \dot\gamma^i X^j = 0.$$

The equation is linear in X, so the solution depends linearly on the initial condition $v \in T_{\gamma(0)}M$. By existence and uniqueness for linear ODEs, for each v there is a unique vector field $X(t)$ along γ. Hence the map

$$P_t : v \longmapsto X(t)$$

is linear and bijective, with inverse given by solving the ODE backward in time.

Concerning the isometry property, let X, Y be vector fields on γ s.t. $\nabla_{\dot\gamma} X = \nabla_{\dot\gamma} Y = 0$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle X, Y \rangle = \dot\gamma(\langle X, Y\rangle) = \big\langle \nabla_{\dot\gamma} X, Y \big\rangle + \big\langle X, \nabla_{\dot\gamma} Y \big\rangle = 0,$$

where we have used that for $f(p) := \big\langle X_p, Y_p \big\rangle$ we have $\frac{\mathrm{d}}{\mathrm{d}t}(f \circ \gamma)(t) = \mathrm{d}\gamma\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)(f) = \dot\gamma(t) f$.
The map $t \mapsto \langle X(t), Y(t) \rangle$ lives on $I \subset \mathbb{R}$, and has zero derivative, so we get $\langle X(t), Y(t)\rangle = \langle X(0), Y(0)\rangle$, which is exactly the isometric property. □

In particular, if $\gamma(a) = \gamma(b)$ then $P_t : T_{\gamma(a)}M \to T_{\gamma(b)}M$ is the identity map, so parallel transport along a closed curve is an isometry of the tangent space at the base point.

6 Geodesics and the exponential map

6.1 Geodesics

We define geodesics as curves whose velocity is parallel along the curve.

Definition 6.1 (Geodesic). A geodesic is a curve $\gamma : I \longrightarrow M$ that satisfies

$$\nabla_{\dot\gamma}\dot\gamma = 0.$$

In local coordinates, this yields the *geodesic equations*,

$$\ddot\gamma^k + \Gamma^k_{ij}\dot\gamma^i\dot\gamma^j = 0 \tag{6.1}$$

where Γ^k_{ij} are the Christoffel symbols of the connection.

Remark 6.2. If we reparametrize: $\tilde\gamma = \gamma \circ h$: one can show that $\nabla_{\dot{\tilde\gamma}}\dot{\tilde\gamma} \parallel \dot{\tilde\gamma}$, i.e. there exists a function $\lambda(t)$ such that $\nabla_{\dot{\tilde\gamma}}\dot{\tilde\gamma} = \lambda\dot{\tilde\gamma}$. But we do not necessarily have $\nabla_{\dot{\tilde\gamma}}\dot{\tilde\gamma} = 0$ anymore so the above definition is parametrisation dependent. Any curve which satisfies $\nabla_{\dot{\tilde\gamma}}\dot{\tilde\gamma} = \lambda\dot{\tilde\gamma}$ can always be reparametrised to geodesics.

Consider now an embedded hypersurface S in $\mathbb{R}^n$ with the induced metric, and the Levi-Civita connection ∇ of g. Recall further example 5.35, and (5.4). One can show that on $S \subset \mathbb{R}^n$, the Levi-Civita connection is the projection of the Euclidean connection on $\mathbb{R}^n$ onto TS (to do so, define $\nabla_X Y := \Pi(D_X Y)$ and verify the assumptions of Theorem 5.34). So we have

$$\nabla_{\dot\gamma}\dot\gamma = \Pi(D_{\dot\gamma}\dot\gamma),$$

where Π is the g-orthogonal projection onto the tangent space $T_{\gamma(t)}S$, and where D is Euclidean connection on $\mathbb{R}^n$ as in equation (5.4).
In this setting, a curve $\gamma : (a,b) \longrightarrow S$ is a geodesic iff

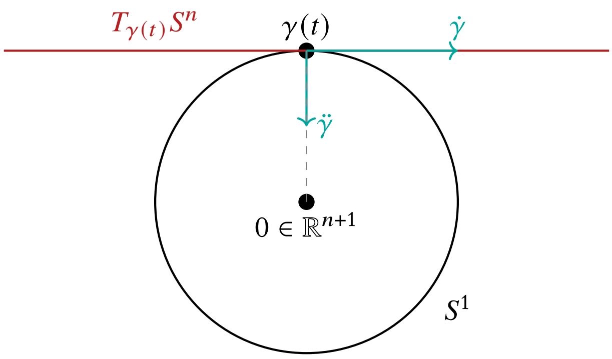
$$0 = \nabla_{\dot\gamma}\dot\gamma = \Pi(D_{\dot\gamma}\dot\gamma),$$

where $\ddot\gamma := D_{\dot\gamma}\dot\gamma = \frac{\mathrm{d}^2\gamma}{\mathrm{d}t^2}$. So here, the *ambient acceleration in* $\mathbb{R}^n$ of a geodesic is orthogonal to the surface, $\ddot\gamma \perp T_{\gamma(t)}S$.

Example 6.3 (Geodesics on (S^n, g_{S^n})). Consider (S^n, g_{S^n}), where g_{S^n} is the metric induced from $(\mathbb{R}^{n+1}, g_E)$. In view of the previous discussion, geodesics γ are exactly curves whose acceleration in $\mathbb{R}^{n+1}$ is orthogonal to TS^n. Since $T_{\gamma(t)}S^n = \gamma(t)^\perp$, the condition $\ddot\gamma \perp T_{\gamma(t)}S^n$ implies $\ddot\gamma = \lambda(t)\gamma$. Hence the subspace

$$V := \operatorname{span}\{\gamma(t), \dot\gamma(t)\}$$

is invariant under time differentiation, so $\gamma(t) \subset V$ for all t. Therefore γ lies in a 2-dimensional linear subspace through the origin, and thus is a great circle.



Theorem 6.4 (Existence and Uniqueness of Geodesics). For every $p \in M$ and $v \in T_pM$, there exists a unique maximal (domain can not be extended) geodesic $\gamma : (-\epsilon, \epsilon) \to M$ such that $\gamma(0) = p$ and $\dot{\gamma}(0) = v$.

For such p, v we will denote the curve by $\gamma_{p,v}(t)$ from now on.

Proof sketch. ODE theory yields existence and uniqueness of solutions to the geodesic equations, (6.1), and that γ depends smoothly on p and v. □

Corollary 6.5. If you have two curves γ_1, γ_2 such that $\gamma_1(0) = \gamma_2(0)$ and $\dot{\gamma}_1(0) = \dot{\gamma}_2(0)$, then on the common domain of intersection they have to agree, i.e. they are subcurves of the same maximal geodesic.

Remark 6.6. Recall that $\Gamma^k_{ij} = 0$ in cartesian coordinates due to (5.13). Hence geodesics satisfy $\ddot{\gamma}^k + \Gamma^k_{ij}\dot{\gamma}^i\dot{\gamma}^j = \ddot{\gamma}^k = 0$, so by integrating twice we get $\gamma^k(t) = a^k t + b^k$, i.e. geodesics are straight lines in $\mathbb{R}^n$. Hence they minimize the distance between points.
On a general Riemannian manifold, this is not necessarily true (for example, on the sphere). However, geodesics are always stationary points of the length functional (Lemma 6.16), and they still minimize length *locally*, i.e. between sufficiently close points (Corollary 6.43).

From the definition, we can easily show that $\|\dot{\gamma}\|^2 = g(\dot{\gamma}, \dot{\gamma})$ is constant in t, i.e. geodesics are parametrised at constant speed.

Lemma 6.7 (Geodesics are parametrised at constant speed). Let γ be a geodesic on (M, g), and let ∇ be the Levi-Civita connection. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} g(\dot{\gamma}, \dot{\gamma}) = 0.$$

Proof.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} g(\dot{\gamma}, \dot{\gamma}) &= \mathrm{d}\gamma\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)(g(\dot{\gamma}, \dot{\gamma})) \\
&= \dot{\gamma}(t) g(\dot{\gamma}, \dot{\gamma}) \\
&= g(\nabla_{\dot{\gamma}}\dot{\gamma}, \dot{\gamma}) + g(\dot{\gamma}, \nabla_{\dot{\gamma}}\dot{\gamma}).
\end{aligned}$$

But $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$ as γ is a geodesic, so we yield the result. □

We present now a version of Nöther's Theorem.

Lemma 6.8 (Nöther's Theorem for Geodesics). Let $X \in \Gamma(TM)$ be Killing, ∇ the Levi-Civity connection, and γ be a Geodesic on (a, b). Then the function $t \mapsto g(X|_{\gamma(t)}, \dot{\gamma})$ is constant for $t \in (a, b)$.

Proof. See exercise 5.1a. □

Any function $F : TM \to \mathbb{R}$ such that $F(\gamma(t), \dot{\gamma}(t))$ is constant when γ is a geodesic is called a *constant of motion* for the geodesic flow. Classically, Nöther's theorem states that for any *symmetry* of the system, there is a corresponding constant of motion. In this case, the function $t \mapsto g(X|_{\gamma(t)}, \dot{\gamma})$ is a constant of motion along the geodesic. The symmetry is given by the Killing vector field X, which, as we have seen is related to isometries of the manifold, see Lemma 5.22.
We will see later, that isometries map geodesics to geodesics.

Example 6.9 (Poincaré Half-Plane, geodesics and constants of motion). Let $\mathbb{H}^2 := \{(x,y) : y > 0\}$ be the Poincaré half-plane with $g_H = \frac{dx^2+dy^2}{y^2}$. In exercise 5.2 we show that

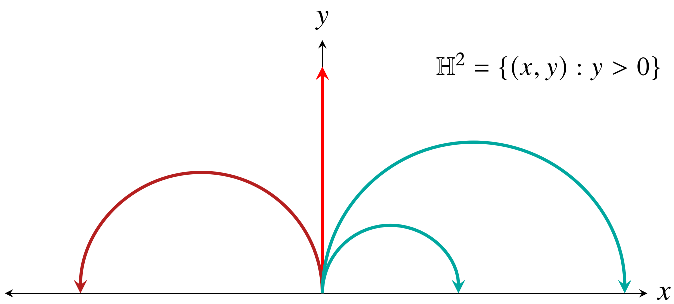
- Maps $f(z) = \frac{az+b}{cz+d}$, with $ad - bc > 0$, are isometries
- The geodesic equations are given by

$$\ddot{x} - \frac{2}{y}\dot{x}\dot{y} = 0, \quad \ddot{y} + \frac{1}{y}(\dot{x}^2 - \dot{y}^2) = 0,$$

  with conserved quantities

$$E = g_H(\dot{\gamma}, \dot{\gamma}), \quad P = \frac{\dot{x}}{y^2},$$

  P being a constant of motion associated to the Killing vector field $\frac{\partial}{\partial x}$, as in Lemma 6.8.
- The geodesics are semicircles orthogonal to the x-axis.



Lemma 6.10 (Parallel orthonormal frame along a geodesic). Let $\gamma : [0,1] \longrightarrow M$ be a geodesic. Then there exists an orthonormal frame $\{E_i\}_{i=1}^n$ along γ such that

$$E_1(t) \text{ is collinear with } \dot{\gamma}(t), \qquad \nabla_{\dot{\gamma}} E_i = 0 \quad \text{for all } i = 1, \ldots, n.$$

Moreover, for any vector field $X = X_i(t)\, E_i|_{\gamma(t)}$ parallel-transported along γ, one has

$$\nabla_{\dot{\gamma}} X = 0 \quad \Longleftrightarrow \quad X_i(t) \text{ is constant in } t \text{ for all } i.$$

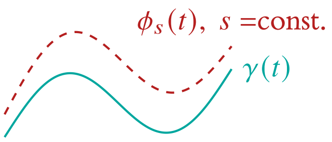
Proof. See exercise 5.3. □

(Lecture 6)

We are now going to investigate the relation between geodesics and distances. We will see that a length minimizing curve is always a geodesic, but a geodesic between two points is *not* always length minimizing.
To talk about length minimizing curves, we need to talk about variations of curves, i.e. how curves vary smoothly.

Definition 6.11 (Smooth Variation of a Curve). A smooth variation of a curve $\gamma : [a,b] \longrightarrow M$ is a smooth map $\phi : [a,b] \times (-\varepsilon, \varepsilon) \longrightarrow M,\ (t,s) \mapsto \phi(t,s)$ such that

$$\phi(t,0) = \gamma(t) \quad \forall t \in [a,b].$$



For $s = 0$ we recover the original curve, but varying s yields other curves. We will use the notation $\phi_s(t) = \phi(t, s)$.

There are two natural vector fields in this situation. For this recall vector fields along a curve from Definition 5.31. We have for a fixed s the *tangent vector field along the variation ϕ*, by

$$\frac{\partial \phi_s}{\partial t} := \mathrm{d}\phi\left(\frac{\partial}{\partial t}\right) \tag{6.2}$$

and the so called *variation vector field*, which is tangent to $s \mapsto \phi_s(t)$ for a fixed t

$$\frac{\partial \phi}{\partial s} := \mathrm{d}\phi\left(\frac{\partial}{\partial s}\right) \tag{6.3}$$

In analogy with vector fields along a curve, $\frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial s}$ cannot always be extended to vector fields on M.

Remark 6.12. Recall chapter 3.5 from [TR24]. Both $t \mapsto \phi$ as well as $s \mapsto \phi$ are maps from an interval to a manifold, so $\frac{\partial}{\partial t}, \frac{\partial}{\partial s}$ are the classic partial derivatives (or tangent vectors in $T\mathbb{R}$), which are then pushed forward by the differentials of ϕ. So (6.2) and (6.3) are defined exactly like the usual tangent vectors of curves on manifolds.
The notation $\frac{\partial \phi}{\partial t}$ is justified because in local coordinates these abstract tangent vectors become the usual partial derivatives.

$$\left(\frac{\partial \phi}{\partial t}\right)^k = \frac{\partial \phi^k}{\partial t}, \qquad \left(\frac{\partial \phi}{\partial s}\right)^k = \frac{\partial \phi^k}{\partial s}$$

where of course ϕ^k are the component functions of the coordinate representation $\widehat{\phi} = \psi \circ \phi = (\phi^1, \ldots, \phi^n)$, where ψ is any chart on the manifold, and $\widehat{\phi} : I \to \mathbb{R}^n$.

Lemma 6.13 (Symmetry of Covariant Derivatives in Variations). Let $\phi : [a, b] \times (-\varepsilon, \varepsilon) \to M$ be a smooth variation of a curve, and ∇ the Levi-Civita connection. Then

$$\nabla_{\frac{\partial \phi}{\partial t}} \frac{\partial \phi}{\partial s} = \nabla_{\frac{\partial \phi}{\partial s}} \frac{\partial \phi}{\partial t}.$$

This means $\left[\frac{\partial \phi}{\partial s}, \frac{\partial \phi}{\partial t}\right] = 0$, where we have to keep in mind that the two fields have to be extended, but this extension does not change the commutator in this case.

Proof. In local coordinates:

$$\left(\nabla_{\frac{\partial \phi}{\partial t}} \frac{\partial \phi}{\partial s}\right)^k = \frac{\partial}{\partial t}\left(\frac{\partial \phi^k}{\partial s}\right) + \Gamma^k_{ij} \frac{\partial \phi^i}{\partial t} \frac{\partial \phi^j}{\partial s}$$

where $\frac{\partial}{\partial t}$ denotes the derivative along the curve $t \mapsto \phi_s(t)$.
Similarly:

$$\left(\nabla_{\frac{\partial \phi}{\partial s}} \frac{\partial \phi}{\partial t}\right)^k = \frac{\partial}{\partial s}\left(\frac{\partial \phi^k}{\partial t}\right) + \Gamma^k_{ij} \frac{\partial \phi^i}{\partial s} \frac{\partial \phi^j}{\partial t}$$

Since $\dfrac{\partial}{\partial s}\dfrac{\partial}{\partial t}\phi^k = \dfrac{\partial}{\partial t}\dfrac{\partial}{\partial s}\phi^k$ and $\Gamma^k_{ij} = \Gamma^k_{ji}$, the two expressions are equal. $\square$

Alternative proof. On the manifold $[a, b] \times (-\varepsilon, \varepsilon)$, we have

$$\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right] = 0.$$

So

$$0 = \mathrm{d}\phi\left(\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right]\right) = \left[\mathrm{d}\phi\left(\frac{\partial}{\partial t}\right), \mathrm{d}\phi\left(\frac{\partial}{\partial s}\right)\right] = \left[\frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial s}\right].$$

Since ∇ is symmetric. The result follows.

□

Lemma 6.14 (Differentiability of Length Function)**.** If γ is a C^1, non-singular ($\dot{\gamma} \neq 0$) curve on (a, b), and ϕ is a smooth variation of γ, the length function $s \mapsto l(\phi_s)$ is differentiable near $s = 0$ and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}s} l(\phi_s) = \int_a^b \frac{\left\langle \nabla_{\frac{\partial \phi}{\partial t}} \frac{\partial \phi}{\partial s}, \frac{\partial \phi}{\partial t} \right\rangle}{\left\| \frac{\partial \phi}{\partial t} \right\|} \mathrm{d}t$$

Proof. Recall Definition 3.5.

$$l(\gamma_s) = \int_a^b \sqrt{\langle \dot{\gamma}_s, \dot{\gamma}_s \rangle} \, \mathrm{d}t = \int_a^b \sqrt{\left\langle \frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial t} \right\rangle} \, \mathrm{d}t.$$

Consequentially,

$$\frac{\mathrm{d}}{\mathrm{d}s} l(\gamma_s) = \int_a^b \frac{\frac{\partial}{\partial s} \left\langle \frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial t} \right\rangle}{2\sqrt{\left\langle \frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial t} \right\rangle}} \mathrm{d}t = \int_a^b \frac{\left\langle \nabla_{\frac{\partial \phi}{\partial s}} \frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial t} \right\rangle}{\left\| \frac{\partial \phi}{\partial t} \right\|} \mathrm{d}t.$$

By the previous lemma 6.13:

$$= \int_a^b \frac{\left\langle \nabla_{\frac{\partial \phi}{\partial t}} \frac{\partial \phi}{\partial s}, \frac{\partial \phi}{\partial t} \right\rangle}{\left\| \frac{\partial \phi}{\partial t} \right\|} \mathrm{d}t$$

□

Theorem 6.15 (First Variation of Length)**.** Let $\gamma : [a, b] \to M$ be smooth with $\|\dot{\gamma}\| = \text{const}$ (I can always reparametrise to achieve this). For any smooth variation of γ, $\phi : [a, b] \times (-\varepsilon, \varepsilon) \to M$,

$$\left.\frac{\mathrm{d}}{\mathrm{d}s} l(\phi_s)\right|_{s=0} = \frac{b-a}{l(\gamma)} \left(\left. \left\langle \frac{\partial \phi}{\partial s}, \dot{\gamma} \right\rangle \right|_{t=a}^{t=b} - \int_a^b \left\langle \left.\frac{\partial \phi}{\partial s}\right|_{s=0}, \nabla_{\dot{\gamma}} \dot{\gamma} \right\rangle \mathrm{d}t \right). \tag{6.4}$$

Proof. Since $\|\dot{\gamma}\| = \text{const}$, we have $l(\gamma) = (b - a)\|\dot{\gamma}\|$.
From the lemma 6.14

$$\left.\frac{\mathrm{d}}{\mathrm{d}s} l(\phi_s)\right|_{s=0} = \int_a^b \frac{\left\langle \nabla_{\dot{\gamma}} \left.\frac{\partial \phi}{\partial s}\right|_{s=0}, \dot{\gamma} \right\rangle}{\|\dot{\gamma}\|} \mathrm{d}t = \frac{1}{\|\dot{\gamma}\|} \int_a^b \left\langle \nabla_{\dot{\gamma}} \left.\frac{\partial \phi}{\partial s}\right|_{s=0}, \dot{\gamma} \right\rangle \mathrm{d}t$$

$$= \frac{1}{\|\dot{\gamma}\|} \int_a^b \left(\frac{\mathrm{d}}{\mathrm{d}t} \left\langle \left.\frac{\partial \phi}{\partial s}\right|_{s=0}, \dot{\gamma} \right\rangle - \left\langle \left.\frac{\partial \phi}{\partial s}\right|_{s=0}, \nabla_{\dot{\gamma}} \dot{\gamma} \right\rangle \right) \mathrm{d}t$$

$$= \frac{1}{\|\dot{\gamma}\|} \left(\left. \left\langle \left.\frac{\partial \phi}{\partial s}\right|_{s=0}, \dot{\gamma} \right\rangle \right|_{t=a}^{t=b} - \int_a^b \left\langle \left.\frac{\partial \phi}{\partial s}\right|_{s=0}, \nabla_{\dot{\gamma}} \dot{\gamma} \right\rangle \mathrm{d}t \right)$$

$$= \frac{b-a}{l(\gamma)} \left(\left\langle \frac{\partial \phi}{\partial s}\bigg|_{s=0}, \dot{\gamma} \right\rangle \bigg|_{t=a}^{t=b} - \int_a^b \left\langle \frac{\partial \phi}{\partial s}\bigg|_{s=0}, \nabla_{\dot{\gamma}}\dot{\gamma} \right\rangle \mathrm{d}t \right).$$

□

Lemma 6.16 (Geodesics are Critical Points of Length). If γ is a geodesic, then γ is a critical point of the length functional l under any variation that fixes the endpoints. That is, if $\phi_s(a) = \gamma(a)$ and $\phi_s(b) = \gamma(b)$ for all s, then $\frac{\mathrm{d}}{\mathrm{d}s} l(\phi_s)\big|_{s=0} = 0$.

Proof. As $\phi_s(a) = \gamma(a)$ and $\phi_s(b) = \gamma(b)$ for all s, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\phi(a,0) = \frac{\mathrm{d}}{\mathrm{d}s}\phi(b,0) = 0.$$

Moreover, since γ is a geodesic, $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$. Hence, by the first variation formula, (6.4),

$$\frac{\mathrm{d}}{\mathrm{d}s} l(\phi_s)\bigg|_{s=0} = 0.$$

□

6.2 Application to classical mechanics

In classical mechanics, we study (say smooth) trajectories of particles $f(t) : [a,b] \to \Omega$, which are curves on a manifold, and we are interested in how these trajectories vary under small perturbations. Here, let $\Omega \subset \mathbb{R}^n$ be an open domain. The variation vector field $\frac{\partial \phi}{\partial s}$ captures exactly this idea of how the trajectory changes when we vary it smoothly. One way to axiomatise the dynamics of a particle is to postulate that the trajectory of the particle makes stationary a certain functional, called the *action*. Let $\mathcal{L} \in C^\infty([a,b] \times \Omega \times \mathbb{R}^n)$. We define the *action*

$$S_{\mathcal{L}}[f] := \int_a^b \mathcal{L}\big(t, f(t), f'(t)\big)\, dt.$$

If f is a stationary point of $S_{\mathcal{L}}$ under a certain class of variations, then f solves a very important PDE, the *Euler–Lagrange equations*, which model the dynamics of the particle.

Lemma 6.17 (First variation and Euler–Lagrange equations). Let $F : (-\delta, \delta) \times [a,b] \to \Omega$ be a smooth variation of f with $F(0,t) = f(t)$ and fixed endpoints $F(s,a) = f(a)$, $F(s,b) = f(b)$. Then we get the *first variation of the action* by

$$\frac{d}{ds} S_{\mathcal{L}}[F(s,\cdot)]\bigg|_{s=0} = \int_a^b \partial_s F^i(0,t) \cdot \left(\partial_{x^i}\mathcal{L} - \frac{d}{dt}\partial_{v^i}\mathcal{L} \right)\big[t, f(t), f'(t)\big] dt.$$

Further, if f is stationary for all such variations, it satisfies the *Euler–Lagrange equations*

$$\partial_{x^i}\mathcal{L}(t,f,f') - \frac{d}{dt}\partial_{v^i}\mathcal{L}(t,f,f') = 0, \quad i = 1, \dots, n. \tag{EL1}$$

Proof. See exercise 7.3. □

Example 6.18 (Particles in a conservative force field). or a particle of mass m moving under a conservative force g with potential $U(x)$ (so that $\nabla U = g$), the Lagrangian is

$$\mathcal{L}(t,x,v) = \frac{1}{2}m|v|^2 - U(x).$$

In this case, the Euler–Lagrange equations reduce to Newton's second law of motion,

$$m\frac{d^2 f^i}{dt^2}(t) = -\partial_i U(f(t)).$$

We place the dynamics on a Riemannian manifold, (M, g). Let $\mathcal{L} : TM \to \mathbb{R}$ be a smooth Lagrangian. For simplicity we consider the general manifold case only with Lagrangians independet of time. For a smooth curve $\gamma : [a, b] \to M$, define the action

$$S_{\mathcal{L}}[\gamma] := \int_a^b \mathcal{L}(\gamma(t), \dot{\gamma}(t))\, dt.$$

Let $\phi : (-\delta, \delta) \times [a, b] \to M$ be a smooth variation of γ with fixed endpoints, and let

$$V(t) := \frac{\partial \phi}{\partial s}(s, t)\bigg|_{s=0}$$

be the variation field along γ.

Lemma 6.19 (First variation and EL on manifolds). The first variation of the action is

$$\frac{d}{ds} S_{\mathcal{L}}[\phi(s, \cdot)]\bigg|_{s=0} = \left[\frac{\partial \mathcal{L}}{\partial v^i}(\gamma, \dot{\gamma})\, V^i\right]_{t=a}^{t=b} + \int_a^b \left(\frac{\partial \mathcal{L}}{\partial x^i}(\gamma, \dot{\gamma}) - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v^i}(\gamma, \dot{\gamma})\right) V^i\, dt.$$

In particular, if γ is a stationary point of $S_{\mathcal{L}}$ under all variations fixing the endpoints, then γ satisfies the Euler–Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial x^i}(\gamma, \dot{\gamma}) - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v^i}(\gamma, \dot{\gamma}) = 0, \quad i = 1, \ldots, n. \tag{EL2}$$

Proof. See exercise 7.4a. □

We remove now all external forces, and study the case where it depends only on position and velocity. This is the case of a free particle moving on a Riemannian manifold. For $(x, v) \in TM$, we choose it be

$$\mathcal{L}(x, v) := \frac{1}{2} g_x(v, v). \tag{6.5}$$

This can be thought of as an extension of the Newtonian function for the kinetic energy in the setting of Riemannian manifolds. For a smooth curve $\gamma : [a, b] \to M$, the associated action is the (curve) *energy*, as investigated in Exercise 2.1, defined by

$$\mathcal{E}[\gamma] := \int_a^b g(\dot{\gamma}(t), \dot{\gamma}(t))dt. \tag{6.6}$$

As we will see, this definition leads to geodesics as the natural trajectories.

Example 6.20 (Energy functional and geodesics). Let $\mathcal{L}$ be as in (6.5), $S = \mathcal{E}$ as in (6.6), and $\phi : (-\delta, \delta) \times [a, b] \to M$ be a smooth variation of γ with fixed endpoints. Then the *first variation of the energy* is

$$\frac{d}{ds}\mathcal{E}[\phi(s, \cdot)]\bigg|_{s=0} = \Big[g\big(\partial_s \phi, \dot{\gamma}\big)\Big]_{t=a}^{t=b} - \int_a^b g\big(\partial_s \phi, \nabla_{\dot{\gamma}}\dot{\gamma}\big)dt.$$

In particular, if γ is stationary under all such variations, then (EL2) reduces to

$$\nabla_{\dot{\gamma}}\dot{\gamma} = 0, \tag{EL3}$$

i.e. γ is a geodesic. See exercise 7.4b for a derivation.

The structure is identical to the Euclidean case: the Euler–Lagrange equations persist in local coordinates, and the choice of pure kinetic energy as the Lagrangian means there are no force terms, and the Euler–Lagrange equations reduce to the geodesic equation.

6.3 The exponential map

Recall that by Theorem 6.4 for any $p \in M$, and $v \in T_pM$ there exists a unique maximal geodesic $\gamma_{p,v}$ with $\gamma_{p,v}(0) = p$, and $\dot{\gamma}_{p,v}(0) = v$, and that the component functions $\gamma^k_{p,v}(t)$ are smooth functions of the initial data p and v in any local coordinate system.
We introduce first the domain of the exponential map. We denote by $\Omega_p \subset T_pM$ the set of tangent vectors $v \in T_pM$ such that $\gamma_{p,v}$ exists for $t \in [0,1]$,

$$\Omega_p = \{\, v \in T_pM : \gamma_{p,v} \text{ exists on } [0,1]\,\}.$$

Lemma 6.21 (Ω_p is star-shaped around the origin)**.** Let $\Omega_p \subset T_pM$ be defined as above. Then Ω_p is *star-shaped* around 0; that is, if $v \in \Omega_p$ and $\lambda \in [0,1]$, then $\lambda v \in \Omega_p$.

Proof. Let $v \in \Omega_p$, so the geodesic $\gamma_{p,v}$ is defined on $[0,1]$. For any $\lambda \in [0,1]$, define $\sigma(t) = \gamma_{p,v}(\lambda t)$. Then σ satisfies the geodesic equation with initial conditions $\sigma(0) = p$ and $\dot{\sigma}(0) = \lambda v$, so by uniqueness $\sigma = \gamma_{p,\lambda v}$. Moreover, σ is defined at least for $t \in [0, 1/\lambda] \supset [0,1]$, which shows that $\gamma_{p,\lambda v}$ exists on $[0,1]$. Therefore, $\lambda v \in \Omega_p$, and since $0 \in \Omega_p$, it follows that Ω_p is star-shaped around 0. □

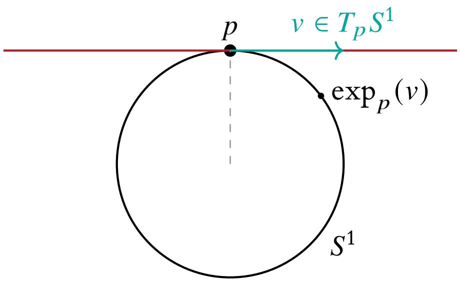
We introduce now the exponential map, which takes a tangent vector and returns the point reached by following the geodesic starting at p in direction v after $t = 1$.

Definition 6.22 (Exponential map)**.** We define $\exp_p : \Omega_p \to M$ such that

$$\exp_p(v) = \gamma_{p,v}(1),$$

with $\gamma_{p,v}$ as in Theorem 6.4. Equivalently, $\exp_p(tv) = \gamma_{p,v}(t)$.

Consider e.g. $M := S^1$, p any point on the circle, and $v \in T_pS^1$ a non-zero tangent vector. Then $\exp_p(v)$ is the point on the circle reached by following the geodesic starting at p in direction v for time $t = 1$.



One can informally think of it as a map sending *straight lines*, i.e. the set $\{\lambda v : \lambda \in [0,1]\}$, through the origin in T_pM to geodesics through p in M.
We study now the differential of the exponential map at v,

$$d\exp_p\big|_v : T_v(T_pM) \to T_{\exp_p(v)}M.$$

For this recall the following.

For any finite dimensional vector space V, we have $V \cong T_v V$ via $\Phi_v : V \to T_v V$. The latter is defined by $u \mapsto D_u|_v$ with $D_u|_v(f) := \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} f(v+tu)$ for any $f \in C^\infty(M)$ (see e.g. Exercise 4.2 of [TR24]). Hence, whenever we view some $u \in T_pM$ as an element of T_vT_pM, we actually work with $\Phi_v(u)$, but write only u. To make this identification be compatible with our Riemannian metric setting, we define an inner product on T_vT_pM via the isomorphism Φ_v as

$$\langle w_1, w_2 \rangle_{T_v(T_pM)} := \langle \Phi_v^{-1}(w_1), \Phi_v^{-1}(w_2) \rangle_{T_pM}.$$

(This is analogous to the pullback of a metric as in Definition 3.12 but Φ_v plays the role of the differential of the inducing map.) With this convention, the inner product on $T_v(T_pM)$ agrees with the inner product on T_pM, under the identification Φ_v.
To give some intuition on the differential of the exponential consider this: By definition, $\exp_p(tv) = \gamma_{p,v}(t)$. We compute now the curve velocity. For this define $c : t \mapsto tv$, so that $\gamma_{p,v} = \exp \circ c$.

$$\begin{aligned}\dot\gamma_{p,v}(t) &= \mathrm{d}\gamma_{p,v}\left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_t\right)\\ &= \mathrm{d}\exp_p|_{c(t)} \circ \mathrm{d}c\left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_t\right)\\ &= \mathrm{d}\exp_p|_{vt}(\dot c(t))\end{aligned}$$

Evaluating at $t = 1$ shows that $\mathrm{d}\exp_p|_v(v)$ is precisely the velocity of the geodesic at $t = 1$, $\dot\gamma_{p,v}(1)$.

Lemma 6.23 (Differential of Exponential Map). The differential $\mathrm{d}\exp_p|_{v=0} : T_vT_pM \to T_pM$ is the identity map, hence $\exp_p$ is a local diffeomorphism around $v = 0$.

Proof. By the inverse function theorem, it suffices to show that $\mathrm{d}\exp_p|_{v=0}$ is invertible. For this recall that $T_{v=0}T_pM \cong T_pM$, as discussed above. We will show that $\mathrm{d}\exp_p|_{v=0} = \mathrm{id}_{T_pM}$. For any map $F : N \to M$, if γ is a curve in N,

$$\mathrm{d}F(\dot\gamma(0)) = \frac{\mathrm{d}}{\mathrm{d}t}(F\circ\gamma)\bigg|_{t=0},$$

see e.g. Proposition 3.17 and Corollary 3.18 in [TR24]. In particular, for $\exp_p$: $t\bar v \mapsto \gamma_{p,\bar v}(t)$, where $\gamma(t) = t\bar v$, $\dot\gamma(0) = \bar v$. So

$$\mathrm{d}\exp_p|_{v=0}(\bar v) = \frac{\mathrm{d}}{\mathrm{d}t}\gamma_{p,\bar v}(t)\bigg|_{t=0} = \bar v \qquad \forall \bar v \in T_pM.$$

Hence $\mathrm{d}\exp_p\big|_{v=0} = \mathrm{id}_{T_pM}$, which is invertible. Therefore, $\exp_p$ is a local diffeomorphism at $v = 0$. □

Lemma 6.24 (Isometries commute with exponential). Let (M, g_M) and (N, g_N) be smooth Riemannian manifolds and let $\Phi : M \to N$ be an isometry, then $\Phi(\exp_p(v)) = \exp_{\Phi(p)}(\mathrm{d}\Phi(v))$ for all $v \in T_pM$.

In other words, the following diagram commutes for any p.

$$\begin{array}{ccc} T_pM & \xrightarrow{d_p\Phi} & T_{\Phi(p)}N \\ \Big\downarrow{\scriptstyle \exp_p} & & \Big\downarrow{\scriptstyle \exp_{\Phi(p)}} \\ M & \xrightarrow[\Phi]{} & N \end{array}$$

Proof. See exercise 6.1. □

A consequence of the above lemma is that if Φ is an isometry, then Φ sends geodesics to geodesics. In particular, if γ is a geodesic in M, then $\Phi \circ \gamma$ is a geodesic in N. This is because $\Phi(\exp_p(v)) = \exp_{\Phi(p)}(\mathrm{d}\Phi(v))$ means that the image under Φ of the geodesic starting at p with initial velocity v is the geodesic starting at $\Phi(p)$ with initial velocity $\mathrm{d}\Phi(v)$. More formally we have

$$\Phi(\gamma_{p,v}(t)) = \gamma_{\Phi(p),\mathrm{d}\Phi(v)}(t)$$

for all $t \in [0, 1]$.

Lemma 6.25 (Isometries and Killing fields on connected manifolds). Let (M, g) be a smooth, *connected* Riemannian manifold.

1. If $\Phi_1, \Phi_2 : M \to M$ are isometries such that, for some $p \in M$, and they agree at p and have the same differential at p, i.e. $\Phi_1(p) = \Phi_2(p)$ and $d_p\Phi_1 = d_p\Phi_2$ then $\Phi_1 = \Phi_2$.
2. If $X \in \Gamma(TM)$ is a Killing vector field such that, for some $p \in M$, $X|_p = 0$, and $(\nabla X)|_p = 0$, then $X = 0$ on M.

Proof. See Exercise 6.2. □

Lemma 6.26 (Geodesics orthogonal to a submanifold). Let (M, g) be a connected Riemannian manifold and $N \subset M$ a smooth submanifold.

1. Let $p \in M$ and $q \in N$ be such that there exists a C^1 curve $\gamma : [0, 1] \to M$ with $\gamma(0) = p$, $\gamma(1) = q$, and $\ell(\gamma) \leq \ell(\sigma)$ for all C^1 curves σ from p to N. Then γ is a geodesic and $\dot\gamma(1) \perp T_qN$.
2. If $\gamma : [0, 1] \to N$ minimizes length between points in N, then $\nabla_{\dot\gamma}\dot\gamma(t) \perp T_{\gamma(t)}N$ for all $t \in [0, 1]$.

Proof. See Exercise 6.3. □

Definition 6.27 (Injectivity Radius). Let $B_r(0) = \{v : \|v\|_g < r\}$. The injectivity radius at $p \in M$ is defined by

$$i(p) = \sup\{r > 0 : \exp_p \text{ is a diffeomorphism on } B_r(0) \subset T_pM\}.$$

The injectivity radius of M is $i(M) = \inf_{p\in M} i(p)$. This is the largest radius for which the exponential map is simultaneously a diffeomorphism at every point on the manifold.
To get some intuition of this slightly abstract definition, we will look at some examples.

Example 6.28. Recall that by Remark 6.6 geodesics on $\mathbb{R}^n$ with the Euclidean metric g_E are straight lines, i.e. $\gamma_{p,v}(t) = p + tv$. So $\exp_p(v) = p + v$, which is a global diffeomorphism. Hence, $i(p) = \infty$ for all p.
Upon identifying $T_p\mathbb{R}^n$ with $\mathbb{R}^n$ itself, via the translation map $v \mapsto p + v$, the exponential map becomes the identity.

Example 6.29. On the round sphere S^2 in $\mathbb{R}^3$, $\exp_p$ maps straight lines on T_pS^2 to great circles on S^2. So $i(p) = \pi R$.

6.4 Normal coordinates

(Lecture 7)

Let (M, g) be a Riemannian manifold. $\forall p \in M$, $v \in T_pM$: $\gamma_{p,v}(t)$ is the maximal geodesic from Theorem 6.4 with $\gamma(0) = p$, $\dot\gamma(0) = v$. Recall the exponential map $\exp_p : \Omega_p \subseteq T_pM \to M$ from Definition 6.22, given by $\exp_p(tv) = \gamma_{p,v}(t)$. The exponential map *flattens out* the geometry of the manifold as much as possible. Its inverse is a coordinate chart of the *normal coordinates*.

For this, we identify T_pM with $\mathbb{R}^n$ via a basis, $\{e_1, \ldots, e_n\}$, which is orthonormal with respect to $g|_p$ (so that any $v \in T_pM$ can be expanded as $v = v^i e_i$). From this perspective one can view the normal coordinates as a push-forward of the *cartesian coordinates on the tangent space* via $\exp_p$.

Remark 6.30. The choice of the basis $\{e_i\}$ is not canonical, and different choices will lead to different normal coordinates.

Recall that any orthonormal basis $\{e_i\}$ of T_pM induces an isomorphism $E : \mathbb{R}^n \to T_pM, x \mapsto x^i e_i$, where $x = (x^1, \ldots, x^n)$. Recall further that $\exp_p$ is a local diffeomorphism by Lemma 6.23, so we can restrict $\exp_p$ to a neighborhood V of $0 \in T_pM$ such that $\exp_p$ is a diffeomorphism there. We set $U := \exp_p(V)$ and call it a *normal neighborhood* of p.

Definition 6.31 (Normal coordinates). Let E and U be as above. Then we define the *normal coordinates* on U by the chart $\varphi : U \to \mathbb{R}^n$ as

$$\varphi := E^{-1} \circ \exp_p^{-1}.$$

Remark 6.32. We have that

- On the chart, $x(q) = (x^1(q), \ldots, x^n(q)) = E^{-1}(\exp_p^{-1}(q))$.
- For $v \in T_pM$, $\exp_p(v) \in M$ and so $x(\exp_p(v)) = E^{-1}(\exp_p^{-1}(\exp_p(v)) = (v^1, \ldots, v^n)$.
- The coordinate expression of the exponential, $\widetilde{\exp}_p$, in normal coordinates is the identity map, as
$$\widetilde{\exp}_p := \varphi \circ \exp_p \circ E = E^{-1} \circ \exp_p^{-1} \circ \exp_p \circ E = \mathrm{Id}_{\mathbb{R}^n}.$$
- Geodesics through p are straight lines. Let $\gamma(0) = p, \dot\gamma(0) = v$. Then $\gamma(t) = \exp_p(tv)$. Define the curve $c(t) := \varphi(\gamma(t))$ on the chart. Then $c(t) = E^{-1}(tv) = t(v^1, \ldots, v^n)$.
- The normal coordinates of p are $(0, \ldots, 0)$, because by definition $\exp_p(0) = p$.

Lemma 6.33 (The metric normal coordinates). In normal coordinates centered at p,

$$g_{ij}(0) = \delta_{ij}, \quad \partial_k g_{ij}(0) = 0, \quad \Gamma^k_{ij}(0) = 0.$$

Second derivatives of the metric in normal coordinates do not necessarily vanish at p. Those are related to the *curvature tensor*.

Proof. Since $x^i(\exp_p(v)) = v^i$ by definition, we have $x^i(\exp_p(t \cdot e_j)) = t\,\delta^i_j$. Hence, for all $j = 0, 1, \ldots, n$, the curve $t \mapsto \exp_p(t \cdot e_j)$ is the x^j-coordinate curve and $\frac{\partial}{\partial x^j}$ is the tangent of this curve. Therefore,

$$\left.\frac{\partial}{\partial x^j}\right|_p = \left.\frac{\mathrm{d}}{\mathrm{d}t}(\exp_p(t\,e_j))\right|_{t=0} = \mathrm{d}\exp_p|_{v=0}(e_j) = e_j.$$

So

$$g_{ij}(x(p)) = g_{ij}(0) = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\bigg|_p = g(e_i, e_j)|_p = \delta_{ij},$$

since $\{e_i\}_{i=1}^n$ is an orthonormal basis.
Recall that by Definition 6.22, the curve $\gamma_{p,v}(t) = \exp_p(tv)$ is a geodesic for any v. Therefore, in normal coordinates: $\gamma^k_{p,v}(t) = t \cdot v^k$. So the geodesic equation, $\ddot{\gamma}^k_{p,v}(t) + \Gamma^k_{ij}(\gamma_{p,v}(t))\,\dot{\gamma}^i_{p,v}\,\dot{\gamma}^j_{p,v} = 0$, becomes

$$\Gamma^k_{ij}(\gamma_{p,v}(t))\, v^i v^j = 0 \tag{6.7}$$

and by letting $t = 0$ we have for any v

$$\Gamma^k_{ij}(0)\, v^i v^j = 0.$$

Since $\Gamma^k_{ij}(0)$ is symmetric in i, j, we obtain $\Gamma^k_{ij}(0) = 0$.
And to obtain the second property above we use Definition 5.25 and Theorem 5.34, we find that

$$\partial_k g_{ij} = \partial_k \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle = g_{aj}\,\Gamma^a_{ki} + g_{ai}\,\Gamma^a_{kj}.$$

So $\partial_k g_{ij}(0) = 0$. □

Remark 6.34. Equation (6.7) shows that the Christoffel symbols vanish along the geodesic $\gamma_{p,v}$, for all v. So the *radial* directions are special. (Geodesics are radial lines in normal coordinates, see previous remark.)

$\exp_p|_v$ is a linear map between inner product spaces, so we can ask how it interacts with the inner product.

Lemma 6.35 (Gauss lemma)**.** For all $v \in \Omega_p \subset T_pM$, and for all $w \in T_v\Omega_p$ we have

$$\left\langle \mathrm{d}\exp_p|_v(v),\, \mathrm{d}\exp_p|_v(w) \right\rangle = \langle v, w\rangle.$$

This means of course that

a) $\|\mathrm{d}\exp_p|_v(v)\| = \|v\|$.

b) For all $w \in T_v(T_pM)$ orthogonal to v, we have:

$$\left\langle \mathrm{d}\exp_p|_v(v),\, \mathrm{d}\exp_p|_v(w) \right\rangle = 0.$$

The Gauss lemma does not imply that $\exp_p$ is an isometry. An alternative proof can be found in exercise 7.4c.

Proof. a) Recall that $\exp_p(tv) = \gamma_{p,v}(t)$. So $\dot{\gamma}_{p,v}(t) = \frac{\mathrm{d}}{\mathrm{d}t}(\exp_p(tv)) = \mathrm{d}\exp_p|_{tv}(v)$. We evaluate this at $t = 1$ and get

$$\dot{\gamma}_{p,v}(1) = \mathrm{d}\exp_p|_v(v).$$

But $\gamma_{p,v}$ is a geodesic, so $\|\dot{\gamma}_{p,v}(1)\| = \|\dot{\gamma}_{p,v}(0)\| = \|v\|$.
b) We will use the first variation formula for the length. Without loss of generality, $\|w\| = \|v\|$. This curve goes through v, and $\alpha(0) = v$ and $\alpha'(0) = w$, so it's a circle around the origin in the tangent space.
Let

$$\alpha(s) = \cos(s)\cdot v + \sin(s)\cdot w,$$

so $\alpha(0) = v$, $\dot{\alpha}(0) = w$, and $\|\alpha(s)\| = \|v\|$, since $v \perp w$, $\|v\| = \|w\|$ wrt. g.

If $\phi(t,s) = \exp_p(t \cdot \alpha(s))$, then the variation of the curve $\gamma_{p,v}(t)$, and $\phi_s(t)$ is a geodesic, because it is $\gamma_{p,\alpha(s)}(t)$, by Definition 6.22. By the first variation formula of the length, (6.4), we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\ell(\phi_s)\bigg|_{s=0} = \frac{1}{\ell(\gamma_{p,v})}\left\langle \frac{\partial\phi}{\partial s}\bigg|_{s=0,\,t=1}, \dot\gamma(1)\right\rangle. \quad (*)$$

But $\ell(\phi_s) = \int_0^1 \left\|\frac{\partial\phi}{\partial t}\right\| \mathrm{d}t = \int_0^1 \|v\| \,\mathrm{d}t = \|v\|$, and $\dot\gamma(1) = \mathrm{d}\exp_p|_v(v)$.

$$\frac{\partial\phi}{\partial s}\bigg|_{t=1} = \frac{\partial}{\partial s}(\exp_p(t\cdot\alpha(s)))\bigg|_{t=1} = \mathrm{d}\exp_p|_{\alpha(s)}(\dot\alpha(s)),$$

and therefore

$$\frac{\partial\phi}{\partial s}\bigg|_{s=0,\,t=1} = \mathrm{d}\exp_p|_v(w).$$

So $(*)$ yields $0 = \left\langle \mathrm{d}\exp_p|_v(v), \mathrm{d}\exp_p|_v(w)\right\rangle$. □

The Gauss Lemma implies that in the *radial direction*, the metric coefficients are the same as in the Euclidean case. For this consider the following situation.
Let $v \in V \subset T_pM$, where V is the neighborhood of 0 on which $\exp_p$ is a diffeomorphism. Let $S \in \Gamma(TM)$ be defined on the normal neighborhood $U = \exp_p(V)$ by

$$S(\exp_p(v)) := \mathrm{d}\exp_p|_v(v),$$

or equivalently $S(q) := \mathrm{d}\exp_p|_{\exp_p^{-1}(q)}(\exp_p^{-1}(q))$, where $q = \exp_p(v) \in U$. Let $(x^1,\ldots,x^n)$ be normal coordinates with associated coordinate basis $\left\{\frac{\partial}{\partial x^i}\big|_p\right\}$.
Let $(x^1,\ldots,x^n)$ be normal coordinates centered at p with coordinate basis $\left\{\frac{\partial}{\partial x^i}\big|_p\right\}$. Since $\varphi = E^{-1}\circ \exp_p^{-1}$, we have $x(q) = \varphi(q) = E^{-1}(v)$, and therefore

$$v = x^i \frac{\partial}{\partial x^i}\bigg|_p.$$

Hence

$$S(q) = x^i\,\mathrm{d}\exp_p|_v\left(\frac{\partial}{\partial x^i}\bigg|_p\right).$$

Since $E: \mathbb{R}^n \to T_pM, x \mapsto x^i e_i$ and $\varphi^{-1} = \exp_p \circ E : \mathbb{R}^n \to M$, we have

$$\frac{\partial}{\partial x^i}\bigg|_q = d(\varphi^{-1})|_x(e_i) = d(\exp_p\circ E)|_x(e_i) = d\exp_p|_v\left(\frac{\partial}{\partial x^i}\bigg|_p\right).$$

Therefore,

$$S(q) = x^i\frac{\partial}{\partial x^i}\bigg|_q.$$

(Alternatively, one can compute the coordinate expression $\widetilde{S} = d\varphi\circ S\circ\varphi^{-1}$. At $x = \varphi(q) = E^{-1}(v)$ this yields $\widetilde{S}(x) = x$, hence $\widetilde{S}^i(x) = x^i$.)
Such vector fields are called *radial vector fields*. This is of course a coordinate-dependent property.

Now using the Gauss lemma, the field S, and the fact that $\mathrm{d}\exp_p|_v$ is the identity in normal coordinates, we get

$$\left\langle x^i\frac{\partial}{\partial x^i}, w^i\frac{\partial}{\partial x^i}\right\rangle = \left\langle S(\exp_p(v)), \mathrm{d}\exp_p|_v(w)\right\rangle = \langle v, w\rangle$$

so we find

$$g_{ij}(x)\,x^i w^j = \delta_{ij}x^i w^j.$$

Since this identity holds for all w, the coefficients of w^j must agree, hence

$$g_{ij}(x)\, x^i = x^j. \tag{6.8}$$

So for the *radial* directions, the metric coefficients are the same as in the Euclidean case.

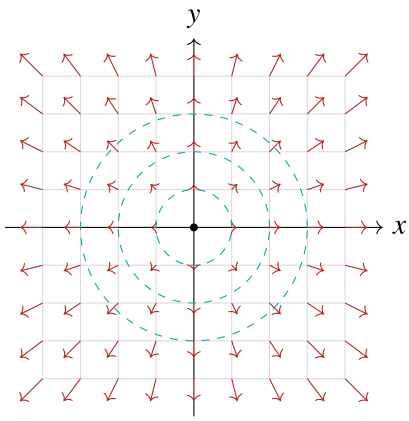


Figure 1: A radial vector field.

Remark 6.36. As also visible in the figure, the radial vector field is orthogonal to the spheres centered at 0 in T_pM. Indeed, one can show that the exponential map sends spheres centered at 0 in T_pM to surfaces orthogonal to the geodesics $\gamma_{p,v}$.

Recall the notion of a *normal neighborhood* from Definition 6.31.

> **Definition 6.37** (Convex neighborhood). We consider normal coordinates on normal neighborhoods U of p (any $q \in U$ can be connected with a geodesic segment to p). The intersection of such neighborhoods is called a *convex* neighborhood.

6.5 Riemannian polar coordinates

Recall remark 6.30. Instead of using Cartesian coordinates on T_pM, we could use polar coordinates and push them forward on M via $\exp_p$.
Assume $\dim M = 2$, and let e_1, e_2 be an orthonormal frame at p. Let $M \supset W_p = \exp_p \left(\{v \in T_pM : \|v\| < i(p)\}\right)$ be the image of the largest ball on which $\exp_p$ is a diffeomorphism, see Definition 6.27 and Lemma 6.23. Let x, y be the normal coordinates of Definition 6.31 on W_p, i.e.

$$x := [E^{-1}(\exp_p^{-1}(q))]^1,$$
$$y := [E^{-1}(\exp_p^{-1}(q))]^2.$$

We define now the polar coordinates induced by the normal coordinates, also known as *Riemannian polar coordinates* by

$$x = r\cos\theta, \quad y = r\sin\theta, \qquad r = \sqrt{x^2+y^2}, \quad \cos\theta = \frac{x}{\sqrt{x^2+y^2}}.$$

In exercise 7.2a, we show that on $\mathbb{R}^n$ they coincide with the usual polar coordinates. In 7.2b and c, we compute metrics on other manifolds in *Riemannian polar coordinates*.

Lemma 6.38 (Metric in polar coordinates). In these coordinates,

$$g = \mathrm{d}r \otimes \mathrm{d}r + b^2(r,\theta)\,\mathrm{d}\theta \otimes \mathrm{d}\theta = \mathrm{d}r^2 + b^2(r,\theta)\,\mathrm{d}\theta^2,$$

with

$$\lim_{r\to 0} b(r,\theta) = 0, \quad \lim_{r\to 0} \frac{\partial b}{\partial r}(r,\theta) = 1, \quad \lim_{r\to 0} \frac{\partial^2 b}{\partial r^2}(r,\theta) = 0.$$

Equivalently, $b(r,\theta) = r + O(r^3)$ as $r \to 0$.

Remark 6.39. On the flat plane $\mathbb{R}^n$, we have $g_E = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2$, which again coincides with the usual polar coordinates. This formula for the metric is derived in exercise 1.3.

Proof. In polar coordinates:

$$q(r,\theta) = \exp_p(r\cos\theta \cdot e_1 + r\sin\theta \cdot e_2) = \exp_p(r \cdot (\cos\theta\, e_1 + \sin\theta\, e_2)).$$

So $r \mapsto q(r,\theta)$ is a geodesic (with θ fixed), and its tangent vector is $\frac{\partial}{\partial r}$. Explicitly:

$$\left.\frac{\partial}{\partial r}\right|_{\exp_p(v)} = d\exp_p\big|_v(\cos\theta\, e_1 + \sin\theta\, e_2),$$

$$\left.\frac{\partial}{\partial \theta}\right|_{\exp_p(v)} = d\exp_p\big|_v(r \cdot (-\sin\theta\, e_1 + \cos\theta\, e_2)),$$

where $v = r(\cos\theta\, e_1 + \sin\theta\, e_2)$.
Since $r \mapsto q(r,\theta)$ is a geodesic, $\left\|\frac{\partial}{\partial r}\right\|$ is constant in r. As $r \to 0$:

$$\left\|\frac{\partial}{\partial r}\right\| \to \|\cos\theta\, e_1 + \sin\theta\, e_2\| = 1,$$

so $g_{rr} = 1$ everywhere. (This is Gauss lemma part (a).)
In other words, since $r \mapsto q(r,\theta)$ is a geodesic with initial velocity $\cos\theta\, e_1 + \sin\theta\, e_2$ (a unit vector), it has constant unit speed. Hence $\left\|\frac{\partial}{\partial r}\right\| \equiv 1$, i.e., $g_{rr} = 1$ everywhere, by Gauss lemma part (a).
For the off-diagonal term, note that in T_pM the vectors $\cos\theta\, e_1 + \sin\theta\, e_2$ (radial) and $-\sin\theta\, e_1 + \cos\theta\, e_2$ (tangential) are orthogonal. By Gauss lemma part (b):

$$g_{r\theta} = \left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta} \right\rangle = r \cdot \langle \cos\theta\, e_1 + \sin\theta\, e_2,\ -\sin\theta\, e_1 + \cos\theta\, e_2 \rangle_{T_pM} = 0.$$

Therefore:

$$g = g_{rr}\, dr^2 + 2g_{r\theta}\, dr\, d\theta + g_{\theta\theta}\, d\theta^2 = dr^2 + g_{\theta\theta}\, d\theta^2.$$

Now we compute $b(r,\theta) = \left\|\frac{\partial}{\partial\theta}\right\| = \sqrt{g_{\theta\theta}}$.
From the coordinate transformation $x = r\cos\theta$, $y = r\sin\theta$:

$$\frac{\partial}{\partial r} = \cos\theta\, \frac{\partial}{\partial x} + \sin\theta\, \frac{\partial}{\partial y},$$

$$\frac{\partial}{\partial \theta} = -r\sin\theta\, \frac{\partial}{\partial x} + r\cos\theta\, \frac{\partial}{\partial y}.$$

Therefore:

$$\begin{aligned} g_{\theta\theta} &= r^2\, g\left(-\sin\theta\, \tfrac{\partial}{\partial x} + \cos\theta\, \tfrac{\partial}{\partial y},\ -\sin\theta\, \tfrac{\partial}{\partial x} + \cos\theta\, \tfrac{\partial}{\partial y}\right) \\ &= r^2\left(\sin^2\theta\, g_{xx} - 2\sin\theta\cos\theta\, g_{xy} + \cos^2\theta\, g_{yy}\right). \end{aligned}$$

In normal coordinates centered at p, we have $g_{ij}(0) = \delta_{ij}$ and $\partial_k g_{ij}(0) = 0$, so Taylor expansion gives:

$$g_{xx} = 1 + O(r^2), \quad g_{xy} = O(r^2), \quad g_{yy} = 1 + O(r^2).$$

Substituting:

$$g_{\theta\theta} = r^2 \left(\sin^2\theta(1 + O(r^2)) - 2\sin\theta\cos\theta \cdot O(r^2) + \cos^2\theta(1 + O(r^2))\right) = r^2 + O(r^4).$$

Therefore:

$$b(r,\theta) = \sqrt{g_{\theta\theta}} = \sqrt{r^2 + O(r^4)} = r + O(r^3),$$

which gives $\lim_{r\to 0} b(r,\theta) = 0$, $\lim_{r\to 0} \frac{\partial b}{\partial r}(r,\theta) = 1$, and $\lim_{r\to 0} \frac{\partial^2 b}{\partial r^2}(r,\theta) = 0$. □

So of course this lemma was just an application of the Gauss lemma rephrased in polar coordinates. but we see that in polar coordinates, the theorem is particularly useful, because the radial direction is special in this setting.

In *Riemannian polar coordinates*, the curve $\gamma(r) = (r,\theta)$ is a geodesic, with

$$\|\dot\gamma\| = \left\|\frac{\partial}{\partial r}\right\| = 1.$$

This is because in the original definition, this curve is $r \mapsto \exp_p(r \cdot u)$ where $u = \cos\theta_0\, e_1 + \sin\theta_0\, e_2$ is a unit vector, which is exactly the geodesic $\gamma_{p,u}$ with initial velocity u. The tangent vector is $\frac{\partial}{\partial r}$, and we just proved $g_{rr} = \left\|\frac{\partial}{\partial r}\right\|^2 = 1$. So indeed $\|\dot\gamma\| = 1$. Therefore, the curve length of γ is

$$\ell(\gamma|_{[0,r]}) = \int_0^r \|\dot\gamma(s)\|\, \mathrm{d}s = \int_0^r 1\, \mathrm{d}s = r.$$

The radial coordinate r in *Riemannian polar coordinates* is literally the distance traveled along the radial geodesic from p. So the *sphere of radius* r in T_pM maps under $\exp_p$ to the set of points reached by traveling distance r along radial geodesics from p.
This is a key step toward showing that radial geodesics are actually distance-minimizing (within the injectivity radius). One direction of the relationship between geodesics and length-minimizing curves is easy: by Lemma 6.16, geodesics are critical points of the length functional. The converse, that a length-minimizing curve must be a geodesic, up to reparametrization by arc length, also follows from the first variation formula, by a bump-function argument applied to the integrand. We will record this as Corollary 6.43 below.
What is not automatic, and is the real content of the next theorem, is that within the injectivity radius radial geodesics actually *achieve* the infimum defining the distance:

Theorem 6.40 (Radial geodesics minimize locally). Let $p, q \in (M,g)$ and $d(p,q) < i(p)$. If $\gamma\colon [0,1] \to M$, $\gamma(0) = p$, $\gamma(1) = q$ is such that $\ell(\gamma) = d(p,q)$, then γ is a radial geodesic from p, up to reparametrization.

Proof. We prove the statement in dimension 2. Let $d = d(p,q)$.
Fix polar coordinates (r,θ) around p, defined on the ball $r < i(p)$. In these coordinates,

$$g = \mathrm{d}r^2 + b^2(r,\theta)\, \mathrm{d}\theta^2.$$

For any $t_0 \in [0,1]$ such that $\gamma|_{[0,t_0]}$ lies in the chart, writing $\gamma(t) = (r(t), \theta(t))$:

$$\ell(\gamma|_{[0,t_0]}) = \int_0^{t_0} \sqrt{\dot r^2 + b^2\dot\theta^2}\, \mathrm{d}t \geqslant \int_0^{t_0} |\dot r|\, \mathrm{d}t \geqslant r(t_0) - r(0) = r(t_0). \tag{6.9}$$

Since $\ell(\gamma) = d < i(p)$, the curve cannot escape the chart: if it did, $r(t)$ would reach $i(p)$, forcing $\ell(\gamma) \geqslant i(p)$, a contradiction. So $\gamma([0,1])$ lies in the chart, and applying (6.9) at $t_0 = 1$ gives $d = \ell(\gamma) \geqslant r(1)$. Conversely, the radial geodesic from p to $\gamma(1)$ has length $r(1)$, so $d \leqslant r(1)$. Hence $\ell(\gamma) = r(1)$, i.e. equality holds in (6.9).
In particular, at every t where γ is differentiable,

$$\sqrt{\dot{r}^2 + b^2(r,\theta)\dot{\theta}^2} = |\dot{r}|,$$

so $b(r,\theta)\dot{\theta} = 0$. Since $b > 0$ for $r > 0$, we get $\dot{\theta}(t) = 0$ for $t > 0$. Thus θ is constant along γ, so γ is a radial curve. Reparametrizing by arc length (i.e. parametrise so that $\|\dot{\gamma}(t)\|$ is constant in t) gives a radial geodesic from p. □

Let (M, g) be a Riemannian manifold and $p \in M$. For any $R < i(p)$, the metric ball and the image of the Euclidean ball under the exponential map coincide.

Corollary 6.41 (Metric balls via the exponential map). We have

$$B(p,R) = \exp_p\{v \in T_pM : \|v\| \leqslant R\}.$$

Moreover, the metric sphere equals the exponential image of the Euclidean sphere:

$$\partial B(p,R) = \exp_p\{v \in T_pM : \|v\| = R\} = \{q \in M : d(p,q) = R\}.$$

The content of this theorem is that two a priori different notions of "ball" agree within the injectivity radius:

- The *metric ball* $B(p,R) = \{q \in M : d(p,q) \leqslant R\}$, defined via the Riemannian distance d, which is an infimum over lengths of *all* curves from p to q.
- The *exponential image* $\exp_p\{v : \|v\| \leqslant R\}$, the set of endpoints of radial geodesics from p of length at most R.

Proof sketch. The inclusion "$\supseteq$" is immediate: if $q = \exp_p(v)$ with $\|v\| \leqslant R$, then the radial geodesic from p to q has length $\|v\|$, so $d(p,q) \leqslant \|v\| \leqslant R$.
For the nontrivial inclusion "$\subseteq$", let $q \in B(p,R)$, so $d(p,q) \leqslant R < i(p)$. Curves γ from p to q with $\ell(\gamma) \geqslant i(p)$ cannot achieve the infimum, so we may restrict attention to curves with $\ell(\gamma) < i(p)$. For any such curve, the argument in the proof of Theorem 6.40 shows that γ stays in the normal chart and satisfies

$$\ell(\gamma) \geqslant \|\exp_p^{-1}(q)\|.$$

Hence the radial geodesic from p to q, which has length exactly $\|\exp_p^{-1}(q)\|$, achieves the infimum, and $d(p,q) = \|\exp_p^{-1}(q)\|$.
By Theorem 6.40, this minimizer is (a reparametrization of) a radial geodesic, so $q = \exp_p(v)$ for some v with $\|v\| = d(p,q) \leqslant R$.
The condition $R < i(p)$ is essential: beyond the injectivity radius, $\exp_p$ fails to be injective, and the exponential image of a Euclidean ball need no longer coincide with the metric ball (e.g., on S^2 with $i(p) = \pi$, the exponential image of a ball of radius $R > \pi$ wraps around the sphere). □

Definition 6.42 (Locally length-minimizing curve). A curve $\gamma\colon I \to (M,g)$ *locally minimizes the length* if for all $t_0 \in I$, there exists $\varepsilon > 0$ such that for all $t_0 - \varepsilon \leqslant t_1 \leqslant t_2 \leqslant t_0 + \varepsilon$:

$$d(\gamma(t_1), \gamma(t_2)) = \ell(\gamma|_{[t_1,t_2]}).$$

We saw earlier the variational fact that geodesics are critical points of the length functional (Lemma 6.16). The corollary below upgrades this to a genuine *minimization* statement, by applying Theorem 6.40 locally; at each point one identifies distance-realizing curves as radial geodesics from that point.

> **Corollary 6.43 (Geodesics minimze the length locally).** A curve locally minimizes the length if and only if it is a reparametrization of a geodesic.

Proof sketch. " $\Longleftarrow$ ": Let γ be a geodesic and $t_0 \in I$. Choose $\varepsilon > 0$ small enough that $\gamma([t_0-\varepsilon, t_0+\varepsilon])$ lies inside a normal ball around $\gamma(t_0)$ of radius less than $i(\gamma(t_0))$. For any t_1, t_2 in this interval, Theorem 6.40 applied at $\gamma(t_1)$ shows that the length-minimizer from $\gamma(t_1)$ to $\gamma(t_2)$ is a radial geodesic.
By uniqueness of geodesics with given initial data (Theorem 6.4), this radial geodesic coincides with $\gamma|_{[t_1,t_2]}$, up to reparametrization. Hence $\ell(\gamma|_{[t_1,t_2]}) = d(\gamma(t_1), \gamma(t_2))$.
" $\Longrightarrow$ ": Suppose γ locally minimizes length. Reparametrize by arc length (i.e. parametrise so that $\|\dot\gamma(t)\|$ is constant in t). For each $t_0 \in I$, pick ε as above so that locally γ realizes the distance between its endpoints. By Theorem 6.40, $\gamma|_{[t_0-\varepsilon,t_0+\varepsilon]}$ is a radial geodesic from $\gamma(t_0-\varepsilon)$, hence in particular a geodesic. Since being a geodesic is a local condition, γ is a geodesic on all of I. □

7 Completeness of manifolds

(Lecture 8)

What can we say about the geodesic flow on Riemannian manifolds that are complete as metric spaces? In what follows, (M, g) is connected.

> **Definition 7.1 (Completeness).** Let (M, g) be a Riemannian manifold.
>
> 1. We will say that a maximal geodesic $\gamma\colon I \to M$ is *complete* if $I = \mathbb{R}$.
> 2. We will say that (M, g) is *geodesically complete* if every maximal geodesic is complete.
> 3. (M, d_g) is a *complete metric space* (or simplify *complete*) if every Cauchy sequence converges.

Examples
- On $(\mathbb{R}^2, g_E)$, all maximal geodesics are complete
- On the disk $(\mathbb{D}^2, g_E)$ this is not true

We will now characterize geodesic completeness (i.e. the Riemannian metric structure) in terms of the metric space structure of (M, d_g). For simplicity we will assume connectedness, but this is not really necessary.

> **Theorem 7.2 (Hopf–Rinow).** The following are equivalent for a connected (M, g):
>
> (a) (M, g) is geodesically complete.
>
> (b) There exists $p \in M$ such that every maximal geodesic through p is complete.
>
> (c) Every closed metric ball in the metric space (M, d_g) is compact.
>
> (d) (M, d_g) is a complete metric space.

Recall that (c) is also called the *Bolzano–Weierstrass property*. Sometimes this theorem is stated only as an equivalence between (a) and (d). The requirement of connectedness is only required for (b).

We will prove first the following intermediate result:

Lemma 7.3 (Geodesics from a complete point). Let (M, g) be connected and $p \in M$. If every maximal geodesic through p is complete, then for all $q \in M$ there exists a geodesic γ from p to q of length $\ell(\gamma) = d(p, q)$.

Remark 7.4. The hypothesis is equivalent to saying that $\Omega_p = T_pM$. As a corollary: if $\Omega_p = T_pM$, then $\exp_p$ is onto.

Locally this is always true, as we have seen before in Lemma 6.23. And it is also globally true, if every maximal geodesic through p is complete.

Proof. Let $d := d(p, q) > 0$.
Choose ε such that $0 < \varepsilon < \min\{i(p),\ d\}$, so that $\exp_p$ is a diffeomorphism on $\overline{B_\varepsilon(0)} \subset T_pM$. By Corollary 6.41,

$$\bar{B}(p, \varepsilon) = \{x \in M : d(p, x) \leqslant \varepsilon\} = \exp_p\{v \in T_pM : \|v\| \leqslant \varepsilon\},$$

and in particular the metric sphere $\partial B(p, \varepsilon) = \exp_p\{v : \|v\| = \varepsilon\}$ is compact.

We choose now the initial direction. Since $\partial B(p, \varepsilon)$ is compact and $x \mapsto d(x, q)$ is continuous, there exists $y_1 \in \partial B(p, \varepsilon)$ minimizing $d(\cdot, q)$ on $\partial B(p, \varepsilon)$. Write $y_1 = \exp_p(v)$ with $\|v\| = \varepsilon$, set $v_1 = v/\|v\|$, and define

$$\gamma(t) := \exp_p(tv_1).$$

By the hypothesis (every maximal geodesic through p is complete), γ is defined for all $t \in \mathbb{R}$. Note that $\|\dot\gamma\| \equiv 1$ and $\gamma(\varepsilon) = y_1$. *Note that this is the only point in the proof where the completeness hypothesis is used.* We claim that $\gamma(d) = q$, which proves the theorem since $\ell(\gamma|_{[0,d]}) = d$.

Step 1: we show that $d(y_1, q) = d - \varepsilon$.
Any piecewise C^1 curve σ from p to q must cross $\partial B(p, \varepsilon)$ at some point y, so

$$\ell(\sigma) \geqslant \varepsilon + d(y, q) \geqslant \varepsilon + d(y_1, q).$$

Taking the infimum over all such σ gives $d \geqslant \varepsilon + d(y_1, q)$, i.e., $d(y_1, q) \leqslant d - \varepsilon$. Combined with the triangle inequality $d \leqslant \varepsilon + d(y_1, q)$, we get equality.

Step 2: Setting up the continuity argument.
By the triangle inequality, for all $s \in [0, d]$,

$$d(p, q) \leqslant d(p, \gamma(s)) + d(\gamma(s), q) \leqslant s + d(\gamma(s), q),$$

so

$$d(\gamma(s), q) \geqslant d - s \qquad \forall\, s \in [0, d]. \tag{7.1}$$

Define

$$J := \{s \in [0, d] : d(\gamma(s), q) = d - s\}.$$

We will show $d \in J$, which gives $d(\gamma(d), q) = 0$, hence $\gamma(d) = q$. We observe:

1. $0 \in J$, since $\gamma(0) = p$.
2. J is downward closed: if $t_0 \in J$ and $t \in [0, t_0]$, then

   $$d(\gamma(t), q) \leqslant d(\gamma(t), \gamma(t_0)) + d(\gamma(t_0), q) \leqslant (t_0 - t) + (d - t_0) = d - t,$$

   which combined with (7.1) gives $t \in J$.

3. J is closed in $[0, d]$, by continuity of $s \mapsto d(\gamma(s), q)$.
4. $\varepsilon \in J$, by Step 1.

Step 3: Inductive step.
We claim that if $t_0 \in J$ with $t_0 < d$, then there exists $\varepsilon' > 0$ such that $t_0 + \varepsilon' \in J$. Choose ε' such that $0 < \varepsilon' < \min\{i(\gamma(t_0)), d(\gamma(t_0), q)\}$. As in Step 1, applied now at $\gamma(t_0)$ instead of p (using that $d(\gamma(t_0), q) = d - t_0 > 0$), there exists $z \in \partial B(\gamma(t_0), \varepsilon')$ minimizing $d(\cdot, q)$ on this sphere, and

$$d(z, q) = d(\gamma(t_0), q) - \varepsilon' = d - t_0 - \varepsilon'.$$

Write $z = \exp_{\gamma(t_0)}(w)$ with $\|w\| = \varepsilon'$, and define

$$\tilde{\gamma}(s) := \exp_{\gamma(t_0)}\left(s \tfrac{w}{\|w\|}\right), \qquad s \in [0, \varepsilon'],$$

so $\tilde{\gamma}(0) = \gamma(t_0)$, $\tilde{\gamma}(\varepsilon') = z$, and $\|\dot{\tilde{\gamma}}\| \equiv 1$. Consider the concatenated curve

$$a(t) := \begin{cases} \gamma(t), & 0 \leqslant t \leqslant t_0, \\ \tilde{\gamma}(t - t_0), & t_0 \leqslant t \leqslant t_0 + \varepsilon'. \end{cases}$$

Then a is piecewise C^1 with $\ell(a) = t_0 + \varepsilon'$, and

$$d(p, z) \;\geqslant\; d(p, q) - d(z, q) \;=\; d - (d - t_0 - \varepsilon') \;=\; t_0 + \varepsilon' \;=\; \ell(a).$$

Since also $d(p, z) \leqslant \ell(a)$ trivially, a is length-minimizing from p to z. By Corollary 6.43, a is (a reparametrization of) a geodesic; since $\|\dot{a}\| = 1$ on each piece, a is itself a geodesic. In particular a is smooth at t_0, so $\dot{a}(t_0^-) = \dot{a}(t_0^+)$. But $a|_{[0,t_0]} = \gamma|_{[0,t_0]}$ gives $\dot{a}(t_0^-) = \dot{\gamma}(t_0)$, so $\tilde{\gamma}(\,\cdot\,)$ and $\gamma(t_0 + \,\cdot\,)$ are geodesics with the same initial position $\gamma(t_0)$ and the same initial velocity $\dot{\gamma}(t_0)$. By uniqueness of geodesics (Theorem 6.4)

$$\tilde{\gamma}(s) = \gamma(t_0 + s) \qquad \forall\, s \in [0, \varepsilon'].$$

In particular $\gamma(t_0 + \varepsilon') = \tilde{\gamma}(\varepsilon') = z$, so

$$d(\gamma(t_0 + \varepsilon'), q) = d(z, q) = d - (t_0 + \varepsilon'),$$

which means $t_0 + \varepsilon' \in J$.

Step 4: Conclusion.
Let $t^* := \sup J$. Since J is closed (item 3), $t^* \in J$. If $t^* < d$, then by Step 3 there exists $\varepsilon' > 0$ with $t^* + \varepsilon' \in J$, contradicting the definition of t^*. Hence $t^* = d$, so $d \in J$ and $\gamma(d) = q$. □

We prove now the Hopf–Rinow theorem.

Proof. $a \implies b$: Obvious, since (a) implies in particular that all maximal geodesics through any fixed p are complete.
$b \implies c$: We first show that $\bar{B}(p, R)$ is compact for the specific p from hypothesis (b). By Lemma 7.3, every $q \in M$ can be written as $q = \exp_p(v)$ with $\|v\| = d(p, q)$, so

$$\bar{B}(p, R) = \exp_p\{v \in T_pM : \|v\| \leqslant R\}.$$

The right hand side is the image, under the continuous map $\exp_p$ (defined on all of T_pM by hypothesis (b)), of a closed and bounded, and therefore compact subset of $T_pM \cong \mathbb{R}^n$. So $\bar{B}(p, R)$ is compact. For an arbitrary $z \in M$ and $R > 0$, the triangle inequality gives

$$\bar{B}(z, R) \subseteq \bar{B}(p, R + d(p, z)).$$

So $\bar{B}(z, R)$ is a closed subset of a compact set, hence compact.
$c \implies d$: Let (x_n) be a Cauchy sequence in M. Then (x_n) is bounded, so it is contained in some closed ball $\bar{B}(x_1, R)$, which is compact by (c). Hence (x_n) has a convergent subsequence $(x_{n_k}) \to x$, and a Cauchy sequence with a convergent subsequence converges to the same limit.
$d \implies a$: Let $\gamma\colon I \to M$ be a maximal geodesic. By Lemma 6.7, $\|\dot\gamma\|$ is constant; reparametrizing by a constant if necessary (which does not change whether $I = \mathbb{R}$), we may assume $\|\dot\gamma\| = 1$. Write $I = (t_-, t_+)$ and assume for contradiction that $t_+ < +\infty$ (the case $t_- > -\infty$ is symmetric). Pick any $0 \in I$. For $t_n \in [0, t_+)$ with $t_n \to t_+$, we have

$$d(\gamma(t_m), \gamma(t_n)) \leqslant |t_m - t_n|,$$

so $(\gamma(t_n))$ is Cauchy and by (d) converges to some $q \in M$. Now apply local ODE existence to the geodesic equation in a chart around q: there exists $\delta > 0$ such that for any initial condition (p_0, v_0) in some neighborhood of $(q, \cdot)$ in the tangent bundle, the geodesic γ_{p_0,v_0} exists on $[0, \delta]$. (Uniformity is possible because the relevant set of initial conditions has compact closure in TM: $\gamma(t_n) \to q$ and $\|\dot\gamma(t_n)\| \equiv 1$.) Pick n large enough that $t_+ - t_n < \delta$ and $(\gamma(t_n), \dot\gamma(t_n))$ lies in this neighborhood. Then the geodesic with initial data $(\gamma(t_n), \dot\gamma(t_n))$ exists on $[0, \delta]$, and by uniqueness it agrees with $\gamma(t_n + \cdot)$ on $[0, t_+ - t_n)$. This extends γ past t_+, contradicting maximality. □

We remark that the term *synthetic geometry* describes the approach using properties like geodesics, distance, etc. Hopf–Rinow is part of a synthetic view of geometry.

8 Curvature

(Lecture 9)

We will study the curvature of a Riemannian manifold from an intrinsic viewpoint: relative acceleration of nearby geodesics, angles of triangles, etc.

Definition 8.1 (Riemann curvature tensor). The *Riemann curvature tensor* $R \in T^1_3(M)$ of a connection ∇ is defined as

$$R(X, Y)\, Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

One can view $R(X, Y)$ as an operator from $\Gamma(TM)$ to itself via $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, so $R(X, Y)\, Z \in \Gamma(TM)$.

Theorem 8.2 (R is a tensor field). R is C^∞-linear in all its arguments.

Proof. It suffices to prove it is C^∞-linear in all its arguments.
Recall: $[fX, Y] = f[X, Y] - Y(f)\, X$.
So

$$\begin{aligned} R(fX, Y)\, Z &= \nabla_{fX} \nabla_Y Z - \nabla_Y \nabla_{fX} Z - \nabla_{[fX,Y]} Z \\ &= f\, \nabla_X \nabla_Y Z - \nabla_Y (f\, \nabla_X Z) - \nabla_{f[X,Y] - Y(f) X} Z \\ &= f\, \nabla_X \nabla_Y Z - Y(f)\, \nabla_X Z - f\, \nabla_Y \nabla_X Z - f\, \nabla_{[X,Y]} Z + Y(f)\, \nabla_X Z \\ &= f\, R(X, Y)\, Z. \end{aligned}$$

Similarly for the other arguments. □

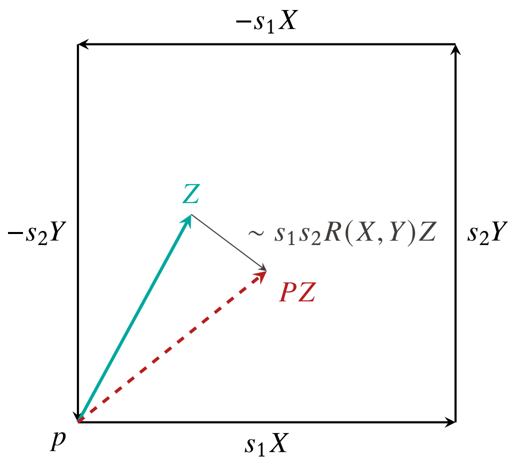
We will be only interested in the Riemann curvature tensor of the Levi-Civita connection of (M, g). To get some geometrical intuition of R consider the following situation.

Let X, Y be coordinate vector fields, so $[X, Y] = 0$, by (5.1). Starting from $p \in M$ and successively flowing along X, Y, $-X$, $-Y$ for small times s_1, s_2, s_1, s_2 respectively, we obtain a closed loop $\gamma(s_1, s_2)$ at p (the loop closes exactly because the flows of X and Y commute). Let $P\colon T_pM \to T_pM$ denote parallel transport once around this loop. Then we have the following

Lemma 8.3 (Characterisation of the Riemann curvature tensor).

$$\frac{PZ - Z}{s_1 \cdot s_2} \xrightarrow{s_1, s_2 \to 0} R(X, Y) \cdot Z.$$

Proof. See exercise 9.4b. □



So $R(X, Y)Z$ measures, to leading order, the failure of parallel transport around an infinitesimal coordinate loop of area s_1s_2 to return Z to itself.

Remark 8.4. This gives a heuristic explanation for one of the symmetries of R that we will prove rigorously in Theorem 8.9. Since P is a linear isometry, expanding $\langle PZ, PW\rangle = \langle Z, W\rangle$ to leading order in s_1s_2 forces

$$\langle R(X, Y)Z, W\rangle + \langle Z, R(X, Y)W\rangle = 0,$$

i.e. $R(X, Y)$ is skew-symmetric in its last two arguments.

Recall the musical isomorphism $\flat\colon TM \to T^*M$, $X \mapsto X^\flat = g(X, \cdot)$, with components $X_a = g_{ab}X^b$. We use it to lower the vector-valued output of R to a $(0, 4)$-tensor. (Since $\nabla g = 0$ for the Levi-Civita connection, lowering and raising indices commutes with ∇, which will be convenient later.)

Definition 8.5 (The (0,4) Riemann curvature tensor). We define the $(0, 4)$-Riemann curvature tensor $\widehat{R} \in T_4^0(M)$ by

$$\widehat{R}(X, Y, Z, W) := g\big(X,\ R(Z, W)\,Y\big).$$

Equivalently, using the musical isomorphism $\flat$:

$$\widehat{R}(X, Y, Z, W) = X^\flat\big(R(Z, W)\,Y\big).$$

Be aware, that this tensor is often the subject of confusion: different authors have different sign conventions, and further, the $\widehat{\cdot}$ symbol is often not written explicitly. We will also adopt this convention.

The type of arguments the tensor is evaluated on will make clear which version of the tensor we are using.
The components of R as a $(1,3)$-tensor are defined by

$$R(\partial_\gamma, \partial_\delta)\, \partial_\beta = R^\alpha{}_{\beta\gamma\delta}\, \partial_\alpha,$$

so that $R^\alpha{}_{\beta\gamma\delta} = \big(R(\partial_\gamma, \partial_\delta)\, \partial_\beta\big)^\alpha$. The components of the $(0,4)$-tensor $\widehat{R}$ are defined as usual by evaluation on the coordinate basis,

$$\widehat{R}_{\alpha\beta\gamma\delta} = \widehat{R}(\partial_\alpha, \partial_\beta, \partial_\gamma, \partial_\delta) = g\big(\partial_\alpha,\ R(\partial_\gamma, \partial_\delta)\, \partial_\beta\big) = g_{\alpha\mu}\, R^\mu{}_{\beta\gamma\delta}.$$

From now on we drop the hat and write

$$R_{\alpha\beta\gamma\delta} := g_{\alpha\mu}\, R^\mu{}_{\beta\gamma\delta}. \tag{8.1}$$

Lemma 8.6 (Components of the Riemann tensor). In local coordinates, the components of the Riemann tensor are

$$R^a{}_{b\gamma\delta} = \partial_\gamma \Gamma^a_{\delta b} - \partial_\delta \Gamma^a_{\gamma b} + \Gamma^a_{\gamma\lambda}\, \Gamma^\lambda_{\delta b} - \Gamma^a_{\delta\lambda}\, \Gamma^\lambda_{\gamma b}. \tag{8.2}$$

The components of the lowered tensor $R_{\alpha\beta\gamma\delta} = g_{\alpha\mu}\, R^\mu{}_{\beta\gamma\delta}$ can be written as

$$R_{\alpha\beta\gamma\delta} = \tfrac{1}{2}\left(\partial^2_{\beta\gamma}\, g_{\alpha\delta} - \partial^2_{\alpha\gamma}\, g_{\beta\delta} + \partial^2_{\alpha\delta}\, g_{\beta\gamma} - \partial^2_{\beta\delta}\, g_{\alpha\gamma}\right) + Q(g, \partial g), \tag{8.3}$$

where $Q(g, \partial g)$ is a function of g, its inverse g^{-1}, and first derivatives ∂g, but no higher derivatives.

We will derive the first formula here. Equation (8.3) is a consequence of the first formula and the Christoffel formula, Definition 5.25.

Proof. By the definition of the Riemann tensor (Definition 8.1) and the fact that coordinate vector fields commute, (5.1),

$$R(\partial_\gamma, \partial_\delta)\, \partial_b = \nabla_{\partial_\gamma} \nabla_{\partial_\delta} \partial_b - \nabla_{\partial_\delta} \nabla_{\partial_\gamma} \partial_b.$$

By the definition of the Christoffel symbols (Definition 5.25):

$$\nabla_{\partial_\delta} \partial_b = \Gamma^\lambda_{\delta b}\, \partial_\lambda.$$

Applying ∇_{∂_γ} to both sides and using the Leibniz rule for the connection (Definition 5.23, axiom 3):

$$\begin{aligned} \nabla_{\partial_\gamma}\left(\Gamma^\lambda_{\delta b}\, \partial_\lambda\right) &= \partial_\gamma\left(\Gamma^\lambda_{\delta b}\right)\, \partial_\lambda + \Gamma^\lambda_{\delta b}\, \nabla_{\partial_\gamma} \partial_\lambda \\ &= \partial_\gamma\left(\Gamma^\lambda_{\delta b}\right)\, \partial_\lambda + \Gamma^\lambda_{\delta b}\, \Gamma^a_{\gamma\lambda}\, \partial_a. \end{aligned}$$

Renaming the dummy index $\lambda \to a$ in the first term:

$$\nabla_{\partial_\gamma}\left(\Gamma^\lambda_{\delta b}\, \partial_\lambda\right) = \partial_\gamma\left(\Gamma^a_{\delta b}\right)\, \partial_a + \Gamma^a_{\gamma\lambda}\, \Gamma^\lambda_{\delta b}\, \partial_a.$$

The analogous expression with γ and δ swapped is:

$$\nabla_{\partial_\delta}\left(\Gamma^\lambda_{\gamma b}\, \partial_\lambda\right) = \partial_\delta\left(\Gamma^a_{\gamma b}\right)\, \partial_a + \Gamma^a_{\delta\lambda}\, \Gamma^\lambda_{\gamma b}\, \partial_a.$$

Subtracting:

$$R(\partial_\gamma, \partial_\delta)\, \partial_b = \left(\partial_\gamma \Gamma^a_{\delta b} - \partial_\delta \Gamma^a_{\gamma b} + \Gamma^a_{\gamma\lambda}\, \Gamma^\lambda_{\delta b} - \Gamma^a_{\delta\lambda}\, \Gamma^\lambda_{\gamma b}\right) \partial_a.$$

Reading off the ∂_a-component yields (8.2).

□

Remark 8.7. Note: $R = 0$ on $(\mathbb{R}^n, g_E)$. Conversely, one can show that (M, g) is locally isometric to $(\mathbb{R}^n, g_E)$ if and only if $R = 0$.

Remark 8.8. In normal coordinates around p, by Lemma 6.33, $g_{ij}|_p = \delta_{ij}$ and $\partial g|_p = 0$. So the formula for $R_{\alpha\beta\gamma\delta}$ from (8.3) simplifies at p to

$$R_{\alpha\beta\gamma\delta}\big|_p = \tfrac{1}{2}\Big(\partial^2_{\beta\gamma}\, g_{\alpha\delta} - \partial^2_{\alpha\gamma}\, g_{\beta\delta} + \partial^2_{\alpha\delta}\, g_{\beta\gamma} - \partial^2_{\beta\delta}\, g_{\alpha\gamma}\Big)\Big|_p, \tag{8.4}$$

since the derivative terms ∂g terms vanish at p. Moreover, by the Gauss lemma, the identity (6.8) holds on the entire normal neighborhood (not just at p):

$$g_{ij}(x)\, x^i = x_j.$$

Differentiating three times with respect to $x^\delta, x^\beta, x^\gamma$ and evaluating at p (where $x = 0$ and $\partial g|_p = 0$) yields the cyclic identity

$$\partial^2_{\beta\gamma}\, g_{\alpha\delta}\big|_p + \partial^2_{\gamma\delta}\, g_{\alpha\beta}\big|_p + \partial^2_{\delta\beta}\, g_{\alpha\gamma}\big|_p = 0.$$

Substituting this into (8.4), the formula simplifies to

$$R_{\alpha\beta\gamma\delta}\big|_p = \Big(\partial^2_{\beta\gamma}\, g_{\alpha\delta} - \partial^2_{\alpha\gamma}\, g_{\beta\delta}\Big)\big|_p.$$

Theorem 8.9 (Symmetries of the Riemann curvature tensor).

1. Antisymmetry in the first pair: $R(X,Y)\,Z = -R(Y,X)\,Z$
2. Antisymmetry in the last arguments: $\langle R(X,Y)\cdot W, Z\rangle = -\langle R(X,Y)\cdot Z, W\rangle$
3. Symmetry pairwise: $R(X,Y,Z,W) = R(Z,W,X,Y)$
4. 1^{st} Bianchi identity: $R(X,Y,Z,W) + R(Z,X,Y,W) + R(Y,Z,X,W) = 0$
5. 2^{nd} Bianchi identity: $(\nabla_X R)(Y,Z,V,W) + (\nabla_Z R)(X,Y,V,W) + (\nabla_Y R)(Z,X,V,W) = 0$

Recall (8.1) for the notation of $R_{\alpha\beta\gamma\delta}$: the last three bullets are the $(0,4)$-Riemann curvature tensor, and the first two are the $(1,3)$-Riemann curvature tensor.

Proof. It suffices to prove the identities in one coordinate system for coordinate vector fields. We choose normal coordinates at p. The statements become:

i–iii) $R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma} = R_{\gamma\delta\alpha\beta}$

iv) $R_{\alpha\beta\gamma\delta} + R_{\gamma\alpha\beta\delta} + R_{\beta\gamma\alpha\delta} = 0$

v) $\nabla_\alpha R_{\beta\gamma\delta\varepsilon} + \nabla_\gamma R_{\alpha\beta\delta\varepsilon} + \nabla_\beta R_{\gamma\alpha\delta\varepsilon} = 0$

Identities i–iii) follow immediately from the explicit expression (8.4), and iv) follows from i–iii). For v), since $\Gamma|_p = 0$, we have

$$\nabla_\alpha R_{\beta\gamma\delta\varepsilon}\big|_p = \partial_\alpha R_{\beta\gamma\delta\varepsilon}\big|_p,$$

and the cyclic sum of $\partial_\alpha R_{\beta\gamma\delta\varepsilon}$ at p vanishes by direct computation from (8.4). □

Remark 8.10 (R as a map on bivectors). Since R is antisymmetric in its first two arguments, we can view it as a map on bivectors:

$$R(X,Y) = R_\wedge(X \wedge Y),$$

where $X \wedge Y \in \Lambda^2 T_pM$. (Recall from Section 4.6 that $\wedge$ can be applied to vectors as well as differential forms.) The antisymmetry $X \wedge Y = -Y \wedge X$ in $\Lambda^2 T_pM$ then automatically encodes the antisymmetry $R(X, Y) = -R(Y, X)$.
This is a special case of the following general algebraic fact. Let V, W be vector spaces and $B : V \times V \to W$ a bilinear, alternating map (i.e. $B(X, Y) = -B(Y, X)$ and $B(X, X) = 0$ for any X). Then B *factors through* $\wedge : V \times V \to \Lambda^2 V$: there is a unique linear map $B_\wedge : \Lambda^2 V \to W$ such that

$$B(X, Y) = B_\wedge(X \wedge Y) \qquad \text{for all } X, Y \in V.$$

One can check this by defining $B_\wedge(X \wedge Y) := B(X, Y)$ on simple bivectors and extending by linearity, then verifying well-definedness (the relations defining $\Lambda^2 V$ are respected) and uniqueness (simple bivectors $X \wedge Y$ span $\Lambda^2 V$).

$$\begin{tikzcd} V \times V \arrow[r, "\wedge"] \arrow[dr, "B"'] & \Lambda^2 V \arrow[d, dashed, "B_\wedge"] \\ & W \end{tikzcd}$$

Lemma 8.11 (Geodesic deviation equation). Let (M, g) be a smooth Riemannian manifold and let $\phi : (-\epsilon, \epsilon) \times [0, 1] \to M$ be a smooth map such that, for each $s \in (-\epsilon, \epsilon)$, the curve $\gamma_s = \phi(s, \cdot)$ is a geodesic. Define the vector fields along ϕ

$$T = \mathrm{d}\phi\left(\frac{\partial}{\partial t}\right), \qquad X = \mathrm{d}\phi\left(\frac{\partial}{\partial s}\right).$$

Then

$$\nabla_T \nabla_T X = -R(X, T)\, T.$$

Proof. See exercise 9.3. □

Here, intuitively, X measures the infinitesimal separation between nearby geodesics; thus, the Riemann curvature tensor measures the *relative acceleration* of nearby geodesics.

Definition 8.12 (Flat manifold). A Riemannian manifold (M, g) is called *(locally) flat* if every point $p \in M$ has a neighborhood that is isometric to an open subset of Euclidean space $(\mathbb{R}^n, g_E)$. Equivalently, around each p one can find coordinates in which $g_{ij} = \delta_{ij}$ on a neighborhood of p.

Remark 8.13. This is a *local* condition. Globally, a flat manifold need not look like $\mathbb{R}^n$. The flat torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$ and the flat cylinder $S^1 \times \mathbb{R}$ are both locally flat but globally distinct from $\mathbb{R}^n$.

Lemma 8.14 (Flat $\iff$ $R = 0$). A Riemannian manifold (M, g) is locally flat if and only if $R \equiv 0$.

Proof sketch. If (M, g) is locally flat, then in coordinates where $g_{ij} = \delta_{ij}$ all Christoffel symbols vanish, so $R = 0$ by the formula (8.2). Since R is a tensor (Theorem 8.2), this holds in every coordinate system. The other direction ($R = 0 \implies$ locally flat) is more delicate. It is proved in Exercise 10.4(c) for the shape operator of geodesic spheres. □

Remark 8.15 (Degrees of freedom of the curvature tensor). In 2D, there is only one independent component. This single number is the *Gaussian curvature.*
Using a combinatorial argument, one can show that in higher dimensions, we have more degrees of freedom. One can show that in 3D, we have 6 independent components. In 4D, we have 20 independent components.

8.1 Sectional curvature

Definition 8.16 (Sectional curvature). Let (M, g) be a Riemannian manifold and $\Pi \subseteq T_pM$ a 2-dimensional subspace spanned by linearly independent vectors $X, Y \in T_pM$. The *sectional curvature* associated to Π is

$$K(X,Y) := \frac{R(X,Y,X,Y)}{\|X\|^2\,\|Y\|^2 - \langle X,Y\rangle^2}.$$

Remark 8.17. The numerator $R(X,Y,X,Y)$ is the $(0,4)$-tensor evaluated on (X,Y,X,Y); equivalently, in $(1,3)$-form,

$$R(X,Y,X,Y) = \langle R(X,Y)\,Y,\ X\rangle.$$

The denominator $\|X\|^2\|Y\|^2 - \langle X,Y\rangle^2 = \|X \wedge Y\|^2$ is the squared area of the parallelogram spanned by X and Y, so $K(\Pi)$ is a *ratio of areas* and depends only on the plane Π, not on the choice of spanning vectors. We will verify this independence in the next lemma.

Lemma 8.18 (Sectional curvature). $K(X,Y)$ is independent of the choice of X, Y spanning Π.

Remark 8.19. As K depends only on the plane Π, we write also $K(\Pi)$ instead of $K(X,Y)$.

Proof. Let e_1, e_2 be an orthonormal basis of Π, so that $X = X^1e_1 + X^2e_2$ and $Y = Y^1e_1 + Y^2e_2$. Then due to the symmetries of R one can show that

$$R(X,Y,X,Y) = \left(\det\begin{pmatrix} X^1 & Y^1 \\ X^2 & Y^2\end{pmatrix}\right)^2 R(e_1,e_2,e_1,e_2),$$

and further one can easily compute that

$$\|X\|^2\,\|Y\|^2 - \langle X,Y\rangle^2 = \left(\det\begin{pmatrix} X^1 & Y^1 \\ X^2 & Y^2\end{pmatrix}\right)^2. \qquad \square$$

Remark 8.20 (R as a quadratic form on bivectors). Recall Section 4.6 and Remark 8.10, where the bivectors $X \wedge Y \in \Lambda^2 T_pM$ were introduced and R was viewed as the operator-valued map $R_\wedge$ acting on a *single* bivector. The map $\widehat{R}_\wedge$ defined below is different: it takes *two* bivectors and returns a scalar in $\mathbb{R}$. $\widehat{R}_\wedge$ is the $(0,4)$ analogue of $R_\wedge$, exactly as $\widehat{R}$ is of R. (For the tensors R and $\widehat{R}$ we follow the convention to drop the hat symbol as remarked earlier).
First, note that the natural inner product on Λ^2T_pM is defined by

$$\langle X \wedge Y,\ Z \wedge W\rangle = \langle X,Z\rangle\langle Y,W\rangle - \langle X,W\rangle\langle Y,Z\rangle,$$

so in particular $\|X \wedge Y\|^2 = \|X\|^2\|Y\|^2 - \langle X,Y\rangle^2$, matching the denominator of K. By the antisymmetries of R (Theorem 8.9, items 1 and 2), the curvature tensor reduces to a bilinear map

$$\widehat{R}_\wedge : \Lambda^2T_pM \times \Lambda^2T_pM \to \mathbb{R}, \qquad \widehat{R}_\wedge(X \wedge Y,\ Z \wedge W) := R(X,Y,Z,W),$$

and the pairwise symmetry (item 3) makes $\widehat{R}_\wedge$ a *symmetric* bilinear form on Λ^2T_pM. In this picture the sectional curvature reads

$$K(\Pi) = \frac{\widehat{R}_\wedge(X \wedge Y,\ X \wedge Y)}{\|X \wedge Y\|^2},$$

which looks a bit like a Rayleigh-quotient: it depends only on the bivector $X \wedge Y$ up to scaling, equivalently only on the plane $\Pi = \mathrm{span}(X,Y)$.

We introduce now the Ricci tensor. It controls volume distortion: it measures how volumes of small geodesic balls deviate from the Euclidean volume in the direction v. Equivalently, Ric governs the average behaviour of nearby geodesics: if $\mathrm{Ric}(\dot\gamma,\dot\gamma) > 0$, geodesics neighbouring γ tend on average to converge.

Definition 8.21 (Ricci curvature). The *Ricci tensor* $\mathrm{Ric} \in T_2^0(M)$ is the trace of the curvature operator in the first slot:

$$\mathrm{Ric}(X,Y) := \mathrm{tr}(Z \mapsto R(Z,X)\,Y)\,.$$

In components,

$$\mathrm{Ric}_{\alpha\beta} = R^{\mu}{}_{\alpha\mu\beta} = g^{\mu\nu}\,R_{\nu\alpha\mu\beta}.$$

If $\{e_i\}_{i=1}^n$ is a local orthonormal frame, then

$$\mathrm{Ric}(X,Y) = \sum_{i=1}^{n} R(e_i, X, e_i, Y).$$

The Ricci tensor is symmetric, $\mathrm{Ric}(X,Y) = \mathrm{Ric}(Y,X)$ (see Exercis 9.1a).

Remark 8.22 (There is one unique Ricci tensor). There are three possible contractions of the upper index of $R^a{}_{bcd}$ with one of the lower indices:

- $R^a{}_{acd} = 0$, since this contracts a symmetric tensor ($g^{a\mu}$, used to raise the index) with one antisymmetric in (μ, a) (the antisymmetry of $R_{\mu acd}$ in its first pair).
- $R^a{}_{bad} = \mathrm{Ric}_{bd}$ (this is of course the Ricci tensor).
- $R^a{}_{bca} = -R^a{}_{bac} = -\mathrm{Ric}_{bc}$, (this is the negative Ricci tensor).

So up to sign, there is only one nontrivial contraction of R.

Definition 8.23 (Scalar curvature). The *scalar curvature* $S \in C^\infty(M)$ is the trace of the Ricci tensor with respect to g,

$$S := g^{ij}\,\mathrm{Ric}_{ij}.$$

Equivalently, in a local orthonormal frame $\{e_i\}_{i=1}^n$, we have

$$S = \sum_{i=1}^{n} \mathrm{Ric}(e_i, e_i) = \sum_{i,j=1}^{n} R(e_i, e_j, e_i, e_j).$$

The three curvatures form a hierarchy of decreasing information:

$$\underbrace{R}_{\text{full tensor}} \longrightarrow \underbrace{\mathrm{Ric}}_{\text{symmetric (0,2)-tensor}} \longrightarrow \underbrace{S}_{\text{scalar}}.$$

Each is a contraction of the previous: Ric is a trace of R, and S is a trace of Ric. At each step information is lost, but the resulting object is simpler to work with.

Lemma 8.24 (Curvatures on a space of constant sectional curvature). If (M,g) has constant sectional curvature K, then

$$\mathrm{Ric}_{ij} = (n-1)\,K\,g_{ij}, \quad S = n(n-1)\,K.$$

Proof. See handwritten notes. □

Example 8.25 (Spaces of constant sectional curvature). The round sphere S^n (and its quotients, e.g. $\mathbb{RP}^n$) has constant positive sectional curvature. Hyperbolic space $\mathbb{H}^n$ has constant negative sectional curvature. Euclidean space $\mathbb{R}^n$ has $K = 0$, hence $\mathrm{Ric} = 0$ and $S = 0$.

9 Riemannian submanifolds

We introduce some natural notions for submanifolds, and characterize the geometry of submanifolds in terms of the curvature tensors.
Let N^n be a submanifold of (M^m, g), so locally, for all $p \in N$, there exists a coordinate system $(x^1, \ldots, x^m)$ on $U \subset M$ such that

$$N \cap U = \{p \in M : x^{n+1}(p) = \ldots. = x^m(p) = 0\}.$$

The tangent space of N at p is

$$T_pN = \{v \in T_pM : v^{n+1} = \cdots = v^m = 0\},$$

where v^i are the coordinates of v in the basis ∂_i. There is also the *perpendicular space* to N at p:

$$T_pN^\perp = \{w \in T_pM : w^1 = \cdots = w^n = 0\} = \{w \in T_pM : w \perp T_pN\}$$

where $\perp$ is to be understood with respect to the metric g. We define the orthogonal projections

$$\pi^T : T_pM \to T_pN \tag{9.1}$$
$$\pi^\perp : T_pM \to T_pN^\perp. \tag{9.2}$$

So all X can be decomposed as

$$v = v^T + v^\perp,$$

where $v^T = \pi^T(v)$ and $v^\perp = \pi^\perp(v)$, because $T_pM = T_pN \oplus T_pN^\perp$.

Definition 9.1 (Metric on Riemannian submanifold). We define the *induced metric* (first fundamental form) on N by $\bar{g} := g|_{T_pN}$, i.e. at each $p \in N$, we take the bilinear form $g_p : T_pM \times T_pM \to \mathbb{R}$ and restrict both arguments to vectors in T_pN:

$$\bar{g}_p(u, v) = g_p(u, v), \quad u, v \in T_pN.$$

For any vector fields X, Y on N, we decompose the covariant derivative (computed in M) into tangential and normal components:

$$\nabla_X Y = \pi^T(\nabla_X Y) + \pi^\perp(\nabla_X Y),$$

where the first term is the induced connection on N, and the second one the *second fundamental form*, also denoted by $B(X, Y) = \pi^\perp(\nabla_X Y) \in TN^\perp$.

(Lecture 10)

We collect three properties of vector $X \in \Gamma(TM)$ fields on a manifold M that are *tangent to* a submanifold N (i.e. satisfy $X_p \in T_pN$ for all $p \in N$).

Lemma 9.2. Let $X \in \Gamma(TM)$ be tangent to N. Then:

(a) X is tangent to N if and only if $X(f)\big|_N = 0$ for every $f \in C^\infty(M)$ that is constant on N.

(b) If $X, Y \in \Gamma(TM)$ are both tangent to N, then $[X, Y]$ is tangent to N.

(c) Every vector field on N admits an extension to a vector field on M tangent to N.

Sketch of proof. Locally, around any $p \in N$, we can find coordinates $(x^1, \dots, x^m)$ such that $N = \{x^{n+1} = \dots = x^m = 0\}$. Then $X_0 = X^i \partial_i \in \Gamma(M, N)$ if $X^i\big|_N = 0$ for $i = n+1, \dots, m$. □

Theorem 9.3 ($\bar{\nabla}$ is the Levi-Civita connection of $\bar{g}$). The connection $\bar{\nabla}$ on N defined by $\bar{\nabla}_X Y := \pi^T(\nabla_X Y)$ is the Levi-Civita connection of the induced metric $\bar{g}$.

Proof. Recall that the Levi-Civita connection is characterized by two properties: torsion-freeness and metric compatibility.
Recall that B is the second fundamental form, defined by $B(X,Y) = \pi^\perp(\nabla_X Y)$, so $\nabla_X Y = \bar{\nabla}_X Y + B(X,Y)$.
Let X, Y, Z be vector fields on N, and extend them to vector fields on M tangent to N (Lemma 9.2 (c)).
Then $[X,Y]$ is also tangent to N (Lemma 9.2 (b)), and along N the ambient bracket $[X,Y]$ coincides with the bracket on N.
Torsion-free: Using $\nabla_X Y = \bar{\nabla}_X Y + B(X,Y)$ with $B(X,Y), B(Y,X) \in TN^\perp$,

$$\bar{\nabla}_X Y - \bar{\nabla}_Y X = (\nabla_X Y)^T - (\nabla_Y X)^T = (\nabla_X Y - \nabla_Y X)^T = [X,Y]^T = [X,Y],$$

where the last equality uses that $[X,Y]$ is tangent to N.
Metric-compatible: For X, Y, Z as above, along N:

$$\begin{aligned}
X(\bar{g}(Y,Z)) &= X(g(Y,Z)) && \text{(since } Y, Z \text{ tangent to } N\text{)} \\
&= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) && (\nabla \text{ metric-compatible}) \\
&= g(\bar{\nabla}_X Y + B(X,Y), Z) + g(Y, \bar{\nabla}_X Z + B(X,Z)) \\
&= g(\bar{\nabla}_X Y, Z) + g(Y, \bar{\nabla}_X Z),
\end{aligned}$$

where the $B(X,Y)$ and $B(X,Z)$ terms vanish since they lie in $TN^\perp$ while $Z, Y \in TN$. Since $\bar{\nabla}_X Y, Z$ and $Y, \bar{\nabla}_X Z$ all lie in TN, we can replace g with $\bar{g}$:

$$X(\bar{g}(Y,Z)) = \bar{g}(\bar{\nabla}_X Y, Z) + \bar{g}(Y, \bar{\nabla}_X Z).$$

□

Theorem 9.4 (Properties of the second fundamental form). The second fundamental form $B(X,Y) = \pi^\perp(\nabla_X Y)$ of a submanifold $N \subset (M, g)$ satisfies:

(a) B is $C^\infty(N)$-bilinear, i.e. a $(0,2)$-tensor on N with values in $TN^\perp$.

(b) As a consequence of (a), the value $B(X,Y)|_p$ at $p \in N$ depends only on X_p and Y_p, giving a well-defined map $B_p : T_pN \times T_pN \to T_pN^\perp$.

(c) B is symmetric: $B(X,Y) = B(Y,X)$.

Proof. Let X, Y be vector fields on M tangent to N, and $f \in C^\infty(N)$ (extended arbitrarily to $C^\infty(M)$).

(a) Linearity in the first argument is immediate from $C^\infty(M)$-linearity of ∇ in its first slot: $B(fX, Y) = \pi^\perp(f\,\nabla_X Y) = f\,B(X,Y)$. For the second:

$$B(X, fY) = \pi^\perp(f\,\nabla_X Y + X(f)\,Y) = f\,B(X,Y) + X(f)\,\underbrace{\pi^\perp(Y)}_{=0} = f\,B(X,Y),$$

since Y is tangent to N.

(b) follows from (a) by the standard pointwise-tensor characterization (Theorem 4.25).
(c) Since ∇ is torsion-free and $[X, Y]$ is tangent to N:

$$B(X,Y) - B(Y,X) = \pi^{\perp}(\nabla_X Y - \nabla_Y X) = \pi^{\perp}([X,Y]) = 0. \qquad \square$$

As in the case of surfaces in $\mathbb{R}^3$: the second fundamental form measures the acceleration (in M) of geodesics in N:

Lemma 9.5 (Geodesic acceleration). If γ is a geodesic of $(N, \bar{g})$, then

$$\nabla_{\dot\gamma}\dot\gamma = B(\dot\gamma, \dot\gamma).$$

Proof. By definition of a geodesic in N, $\bar\nabla_{\dot\gamma}\dot\gamma = 0$. The decomposition $\nabla_{\dot\gamma}\dot\gamma = \bar\nabla_{\dot\gamma}\dot\gamma + B(\dot\gamma, \dot\gamma)$ then gives the claim. $\square$

Definition 9.6 (Totally geodesic submanifold). $N \subset (M, g)$ is *totally geodesic* if every geodesic in $(N, \bar{g})$ is also a geodesic in (M, g). Equivalently, by Lemma 9.5, $B \equiv 0$.

Example 9.7. A linear subspace of $\mathbb{R}^n$, or the equator $S^{n-1} \subset S^n$.

9.1 The fundamental equations for a Riemannian submanifold

Theorem 9.8 (Weingarten equation). Let $N \subset (M, g)$ be a submanifold, let X, Y be vector fields on M tangent to N, and let $W \in \Gamma(TM)$ with $W \perp TN$ along N. Then at every point of N,

$$g(\nabla_X W,\ Y) = -\,g(W,\ B(X,Y)).$$

Proof. Along N, $g(W, Y) = 0$. Since X is tangent to N, differentiating along X and using metric compatibility:

$$\begin{aligned} 0 &= X(g(W,Y)) \\ &= g(\nabla_X W, Y) + g(W, \nabla_X Y) \\ &= g(\nabla_X W, Y) + g(W, \bar\nabla_X Y + B(X,Y)) \\ &= g(\nabla_X W, Y) + g(W, B(X,Y)), \end{aligned}$$

where the term $g(W, \bar\nabla_X Y) = 0$ since $\bar\nabla_X Y \in TN$ and $W \perp TN$. $\square$

The second fundamental form B relates the intrinsic curvature of N to the ambient curvature of M:

Theorem 9.9 (Gauss equation). For all $X, Y, Z, W \in T_pN$ at a point $p \in N$:

$$\bar{R}(X,Y,Z,W) - R(X,Y,Z,W) = \langle B(X,Z),\ B(Y,W)\rangle - \langle B(X,W),\ B(Y,Z)\rangle,$$

where $\bar{R}$ is the Riemann curvature tensor of $(N, \bar{g})$ and R that of (M, g).

We will prove this later.

The right-hand side depends only on B, so the discrepancy between the intrinsic and ambient curvatures is entirely controlled by the second fundamental form.

Corollary 9.10 (Gauss equation for totally geodesic submanifolds). If $N \subset (M, g)$ is totally geodesic, then for all $X, Y, Z \in T_pN$ at $p \in N$,

$$\bar{R}(X,Y)\,Z = R(X,Y)\,Z.$$

Hence, intrinsically, N *looks the same* as the ambient M from the curvature point of view.

Corollary 9.11 (Gauss equation for sectional curvature). If $\Pi \subset T_pN$ is a 2-plane spanned by $X, Y \in T_pN$:

$$\bar{K}(\Pi) = K(\Pi) + \frac{\langle B(X,X),\, B(Y,Y)\rangle - \|B(X,Y)\|^2}{\|X \wedge Y\|^2}.$$

Remark 9.12 (Theorema Egregium). For a surface $N^2 \subset \mathbb{R}^3$ with the induced metric, the ambient sectional curvature vanishes ($K \equiv 0$ in Euclidean space), so the Gauss equation reduces to

$$\bar{K}(\Pi) = \frac{\langle B(X,X),\, B(Y,Y)\rangle - \|B(X,Y)\|^2}{\|X \wedge Y\|^2}.$$

This is the Gauss curvature of the surface. Although the right-hand side is defined using the ambient connection on $\mathbb{R}^3$ (since $B = \pi^\perp \circ \nabla$ uses the ambient ∇ and the orthogonal projection), the *Theorema Egregium* from Gauss states that the result depends only on $\bar{g}$, the induced metric!

Definition 9.13 (Hypersurface). A submanifold $N \subset M$ is called a *hypersurface* if $\dim N = \dim M - 1$.

In this case $T_pN^\perp$ is one-dimensional for all $p \in N$.

Definition 9.14 (Co-orientation). A *co-orientation* of a hypersurface $N \subset (M, g)$ is a smooth unit normal vector field $\hat{n}$ along N, i.e. $\hat{n}_p \in T_pN^\perp$ with $\|\hat{n}_p\| = 1$ for all $p \in N$.

Remark 9.15. A co-orientation exists locally around every point, but not always globally: the Möbius strip is an example of a hypersurface with no global co-orientation. When a global co-orientation exists, N is often called *co-orientable*.

Definition 9.16 (Scalar second fundamental form). Let $N \subset (M, g)$ be a co-oriented hypersurface with unit normal $\hat{n}$. The *scalar second fundamental form* is the $(0,2)$-tensor

$$b(X,Y) := g(B(X,Y),\, \hat{n}), \qquad X, Y \in TN,$$

so that $B(X,Y) = b(X,Y)\,\hat{n}$.

There are two more useful equivalent expressions for b:

- Since $B(X,Y) = \pi^\perp(\nabla_X Y)$ and $\hat{n}$ is a unit normal, $\pi^\perp(\nabla_X Y) = g(\nabla_X Y, \hat{n})\,\hat{n}$, hence

$$b(X,Y) = g(\nabla_X Y,\, \hat{n}).$$

- By the Weingarten equation (Theorem 9.8) applied with $W = \hat{n}$,

$$b(X,Y) = -\,g(\nabla_X \hat{n},\, Y). \tag{9.3}$$

(Lecture 11)

We will now prove Theorem 9.9.

Proof. Extend X, Y, Z, W to vector fields on M tangent to N. Without loss of generality, choose extensions with $[X,Y]_p = 0$ at the point of interest (this is always possible by working in normal coordinates centered at p).
Recall that, by the symmetries of R (Theorem 8.9), $R(X,Y,Z,W) = \langle R(X,Y)W,\ Z\rangle$. We expand the intrinsic curvature using $\nabla_X Y = \bar\nabla_X Y + B(X,Y)$:

$$\begin{aligned}\bar\nabla_X\bar\nabla_Y W &= \nabla_X(\bar\nabla_Y W) - B(X,\bar\nabla_Y W)\\ &= \nabla_X(\nabla_Y W - B(Y,W)) - B(X,\bar\nabla_Y W)\\ &= \nabla_X\nabla_Y W - \nabla_X(B(Y,W)) - B(X,\bar\nabla_Y W).\end{aligned}$$

Taking the inner product with $Z \in TN$:

$$\begin{aligned}\langle\bar\nabla_X\bar\nabla_Y W,\ Z\rangle &= \langle\nabla_X\nabla_Y W,\ Z\rangle - \langle\nabla_X(B(Y,W)),\ Z\rangle - \underbrace{\langle B(X,\bar\nabla_Y W),\ Z\rangle}_{=0,\ B\perp TN}\\ &= \langle\nabla_X\nabla_Y W,\ Z\rangle + \langle B(Y,W),\ \nabla_X Z\rangle \quad \text{(Weingarten, Theorem 9.8)}\\ &= \langle\nabla_X\nabla_Y W,\ Z\rangle + \langle B(Y,W),\ B(X,Z) + \underbrace{\bar\nabla_X Z}_{\in TN}\rangle\\ &= \langle\nabla_X\nabla_Y W,\ Z\rangle + \langle B(Y,W),\ B(X,Z)\rangle,\end{aligned}$$

where the last step uses $B(Y,W) \perp TN$. Subtracting the analogous expression with X replaced by Y and using $[X,Y]_p = 0$ to identify

$$\bar\nabla_X\bar\nabla_Y W - \bar\nabla_Y\bar\nabla_X W = \bar R(X,Y)W, \qquad \nabla_X\nabla_Y W - \nabla_Y\nabla_X W = R(X,Y)W,$$

yields the Gauss equation. □

Definition 9.17 (Weingarten map). Let $N \subset (M,g)$ be a co-oriented hypersurface with unit normal $\hat n$. The *Weingarten map* is the linear endomorphism

$$L_p\colon T_pN \to T_pN, \qquad L_p(X) := (\nabla_X\hat n)^T.$$

The Weingarten map is well-defined: $\nabla_X\hat n$ has no normal component since $\|\hat n\| = 1$ implies $\langle\nabla_X\hat n, \hat n\rangle = \frac{1}{2}X(\|\hat n\|^2) = 0$, so $(\nabla_X\hat n)^T = \nabla_X\hat n$.
Combining the Weingarten map with the alternative form of b from (9.3) we obtain

$$b(X,Y) = -\langle\nabla_X\hat n,\ Y\rangle = -\langle L(X),\ Y\rangle.$$

Since b is symmetric (Theorem 9.4 (c)), so is L with respect to $\bar g$:

$$\langle L(X), Y\rangle = \langle X, L(Y)\rangle.$$

Hence L_p is a self-adjoint endomorphism of T_pN and its eigenvalues are real.

Definition 9.18 (Principal, Gauss, and mean curvatures). Let L_p be the Weingarten map at $p \in N$, with eigenvalues $k_1(p), \dots, k_n(p) \in \mathbb{R}$. We call:

- the eigenvalues $k_i(p)$ the *principal curvatures* at p;
- the determinant $K_{\text{Gauss}}(p) := \det L_p = k_1(p)\cdots k_n(p)$ the *Gauss curvature*;
- the normalized trace $H(p) := \frac{1}{n}\operatorname{tr} L_p = \frac{1}{n}(k_1(p) + \dots + k_n(p))$ the *mean curvature*;
- p an *umbilic point* if $k_1(p) = \dots = k_n(p)$.

Special case: surfaces in 3-space. If $N^2 \subset M^3$ is a hypersurface, L_p has two eigenvalues k_1, k_2, so $K_{\text{Gauss}} = k_1 k_2$. The Gauss equation (Corollary 9.11) then reads

$$\bar{K}(p) = K(p) + K_{\text{Gauss}}(p),$$

where $\bar{K}(p)$ is the *intrinsic* sectional curvature of $(N, \bar{g})$ at p and $K(p)$ is the sectional curvature of the ambient M along T_pN. Note that $K_{\text{Gauss}} = \det L_p$ is defined *extrinsically*, via the Weingarten map, so a priori $\bar{K}$ and K_{Gauss} are different quantities. When the ambient space is flat ($K \equiv 0$, e.g. $M = \mathbb{R}^3$), the Gauss equation forces $\bar{K} = K_{\text{Gauss}}$: the extrinsically-defined Gauss curvature turns out to be intrinsic. This is Gauss's *Theorema Egregium*, see Remark 9.12.
Level-set hypersurfaces. If $N \subset M$ is given as a regular level set $\{f = c\}$ with $\mathrm{d}f|_N \neq 0$, then the unit normal and scalar second fundamental form are

$$\hat{n} = \frac{\operatorname{grad} f}{\|\operatorname{grad} f\|}, \qquad b(X,Y) = -\frac{\operatorname{Hess} f(X,Y)}{\|\operatorname{grad} f\|},$$

where $\operatorname{grad} f = (g^{ab}\,\partial_b f)\,\partial_a$ is the metric gradient (Definition 3.20) and

$$\operatorname{Hess} f := \nabla\,\mathrm{d}f$$

is the Hessian, with components $(\operatorname{Hess} f)_{\alpha\beta} = \partial^2_{\alpha\beta} f - \Gamma^k_{\alpha\beta}\,\partial_k f$. These formulas are derived in Exercise 11.1, which also shows that $\operatorname{Hess} f$ is symmetric and that $\operatorname{Hess} f(X,X) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}(f \circ \gamma(t))\big|_{t=0}$ along the geodesic $\gamma(t) = \exp_p(tX)$.
Lie-derivative formula. More generally, one can show that

$$b = -\tfrac{1}{2}\,\mathcal{L}_{\hat{n}}\,g,$$

where $\mathcal{L}_{\hat{n}}$ is the Lie derivative along $\hat{n}$ (extended to a vector field on a neighborhood of N).
We consider now graph hypersurfaces. A graph $\{t = f(x)\}$ is a globally co-oriented hypersurface of Euclidean space, for which the above quantities specialize to explicit formulas. Since the ambient space is flat, the Gauss equation determines the intrinsic curvature entirely from b (a concrete instance of the Theorema Egregium discussed above).

Lemma 9.19 (Curvature of a graph submanifold). Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be smooth and let $M_f = \{(t,x) \in \mathbb{R}^{n+1} : t = f(x)\} \subset (\mathbb{R}^{n+1}, g_E)$ be its graph, with induced metric $\bar{g}$. Then, in the coordinates $(x^1, \dots, x^n)$ on M_f:

$$\begin{aligned}
\bar{g}_{ij} &= \delta_{ij} + \partial_i f\,\partial_j f,\\
b_{ij} &= \frac{\partial_i\partial_j f}{\sqrt{1 + |\nabla f|^2}},\\
\bar{R}_{ijkl} &= \frac{\partial_i\partial_k f\,\partial_j\partial_l f - \partial_i\partial_l f\,\partial_j\partial_k f}{1 + |\nabla f|^2},
\end{aligned}$$

where b_{ij} is the scalar second fundamental form with respect to the coorientation $\hat{n} = (1 + |\nabla f|^2)^{-1/2}(\partial_t - \sum_j \partial_j f\,\partial_{x^j})$, and $|\nabla f|^2 := \sum_i (\partial_i f)^2$.

Proof. See Exercise 10.3. □

10 Jacobi fields and comparison theorems

10.1 Second variation formula

We have already seen (Corollary 6.43) that geodesics are locally length-minimizing: on sufficiently short intervals, a geodesic realizes the distance between its endpoints. However, this fails on long intervals. A great circle on the sphere is a geodesic, but going more than halfway around it no longer minimizes distance.
The second variation of length is the tool that detects this failure: γ is no longer length-minimizing on $[a, b]$ as soon as we can find an endpoint-fixing variation ϕ_s with $\frac{d^2}{ds^2}\ell(\phi_s)|_{s=0} < 0$. We will use this in the Theorems from Bonnet–Myers and from Cartan–Hadamard to deduce topological properties from curvature.

Theorem 10.1 (Second variation of length). Let $\gamma\colon [a, b] \to (M, g)$ be a geodesic parametrized by arc length ($\|\dot\gamma\| = 1$), and let $\phi\colon (-\varepsilon, \varepsilon) \times [a, b] \to M$ be a variation of γ (i.e. $\phi_0(t) = \gamma(t)$). Set $X := \partial\phi/\partial s$, the variation vector field. Then

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\,\ell(\phi_s)\Big|_{s=0} = \langle \nabla_X X,\, \dot\gamma\rangle\Big|_a^b + \int_a^b \Big(\|\nabla_{\dot\gamma} X^\perp\|^2 - R(\dot\gamma, X, \dot\gamma, X)\Big)\,\mathrm{d}t,$$

where $X^\perp := X - \langle X, \dot\gamma\rangle\,\dot\gamma$ is the component of X perpendicular to $\dot\gamma$.

The proof requires a few preliminary identities. Set $T := \partial\phi/\partial t$.

Lemma 10.2 (Commutation identities for variations). With T, X as above:

(a) $[T, X] = 0$, hence $\nabla_T X = \nabla_X T$.

(b) $\langle \nabla_X\nabla_T X,\, T\rangle = \langle \nabla_T\nabla_X X,\, T\rangle - R(T, X, T, X)$.

Proof. **(a)** Since T, X are pushforwards of the commuting coordinate fields $\partial/\partial t, \partial/\partial s$ and using (5.1):

$$[T, X] = \mathrm{d}\phi([\partial/\partial t,\, \partial/\partial s]) = 0.$$

Torsion-freeness of ∇ then gives $\nabla_T X - \nabla_X T = [T, X] = 0$.
(b) By the definition of R and using $[X, T] = 0$:

$$R(X, T)\,X = \nabla_X\nabla_T X - \nabla_T\nabla_X X.$$

Taking the inner product with T and using the identity $R(A, B, C, D) = g(R(A, B)D, C)$ (symmetries of R, Theorem 8.9):

$$\langle \nabla_X\nabla_T X, T\rangle - \langle \nabla_T\nabla_X X, T\rangle = \langle R(X, T)X, T\rangle = R(X, T, T, X) = -R(T, X, T, X). \qquad \square$$

Lemma 10.3 (First derivative of length, general s). Let $\phi : (-\varepsilon, \varepsilon) \times [a, b] \to M$ be a variation with $T := \partial\phi/\partial t$ nonvanishing. Then for all $s \in (-\varepsilon, \varepsilon)$:

$$\frac{\mathrm{d}}{\mathrm{d}s}\,\ell(\phi_s) = \int_a^b \frac{\langle \nabla_T X,\, T\rangle}{\|T\|}\,\mathrm{d}t.$$

Remark 10.4. At $s = 0$, $T = \dot\gamma$, and if $\|\dot\gamma\|$ is constant, integration by parts recovers the first-variation formula (6.4). The unintegrated form above is needed because we will differentiate once more in s to obtain the second variation, which requires an expression valid in a neighborhood of $s = 0$, not just at the single point.

Proof. Differentiating $\ell(\phi_s) = \int_a^b \sqrt{\langle T, T\rangle}\, \mathrm{d}t$ in s:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}\ell(\phi_s) &= \int_a^b \frac{1}{2\sqrt{\langle T, T\rangle}} \frac{\mathrm{d}}{\mathrm{d}s}\langle T, T\rangle\, \mathrm{d}t \\ &= \int_a^b \frac{\langle \nabla_X T,\, T\rangle}{\|T\|}\, \mathrm{d}t \\ &= \int_a^b \frac{\langle \nabla_T X,\, T\rangle}{\|T\|}\, \mathrm{d}t, \end{aligned}$$

using metric compatibility in the second line and Lemma 10.2 (a) in the third. □

Lemma 10.5 (Perpendicular component of $\nabla_{\dot\gamma} X$ at $s = 0$). At $s = 0$:

$$\|\nabla_{\dot\gamma} X^\perp\|^2 = \|\nabla_{\dot\gamma} X\|^2 - \langle \nabla_{\dot\gamma} X,\, \dot\gamma\rangle^2.$$

Proof. At $s = 0$, $T|_{s=0} = \dot\gamma$ and $\|\dot\gamma\| = 1$, so $X^\perp = X - \langle X, \dot\gamma\rangle\, \dot\gamma$. Differentiating along $\dot\gamma$ and using that $\nabla_{\dot\gamma}\dot\gamma = 0$ (geodesic):

$$\nabla_{\dot\gamma} X^\perp = \nabla_{\dot\gamma} X - \langle \nabla_{\dot\gamma} X, \dot\gamma\rangle\, \dot\gamma = (\nabla_{\dot\gamma} X)^\perp,$$

where the last equality is the orthogonal decomposition of $\nabla_{\dot\gamma} X$ with respect to $\dot\gamma$. Taking the squared norm:

$$\|\nabla_{\dot\gamma} X^\perp\|^2 = \|(\nabla_{\dot\gamma} X)^\perp\|^2 = \|\nabla_{\dot\gamma} X\|^2 - \langle \nabla_{\dot\gamma} X, \dot\gamma\rangle^2,$$

where the last equality uses the Pythagorean identity for the decomposition. □

We are now ready to prove Theorem 10.1.

Proof of the theorem. By Lemma 10.3,

$$\frac{\mathrm{d}}{\mathrm{d}s}\ell(\phi_s) = \int_a^b \frac{\langle \nabla_T X,\, T\rangle}{\|T\|}\, \mathrm{d}t.$$

We differentiate the integrand once more in s using the quotient rule. Using metric compatibility and $\frac{\partial}{\partial s}\|T\| = \langle \nabla_X T, T\rangle / \|T\|$:

$$\frac{\partial}{\partial s}\left(\frac{\langle \nabla_T X, T\rangle}{\|T\|}\right) = \frac{\langle \nabla_X \nabla_T X, T\rangle + \langle \nabla_T X, \nabla_X T\rangle}{\|T\|} - \frac{\langle \nabla_T X, T\rangle\langle \nabla_X T, T\rangle}{\|T\|^3}.$$

Now apply Lemma 10.2: part (a) gives $\nabla_X T = \nabla_T X$, and part (b) gives $\langle \nabla_X \nabla_T X, T\rangle = \langle \nabla_T \nabla_X X, T\rangle - R(T, X, T, X)$. Substituting:

$$\frac{\partial}{\partial s}\left(\frac{\langle \nabla_T X, T\rangle}{\|T\|}\right) = \frac{\langle \nabla_T \nabla_X X, T\rangle - R(T, X, T, X) + \|\nabla_T X\|^2}{\|T\|} - \frac{\langle \nabla_T X, T\rangle^2}{\|T\|^3}.$$

Evaluating at $s = 0$, where $T = \dot\gamma$, $\|T\| = 1$, and $\nabla_{\dot\gamma}\dot\gamma = 0$:

- $\langle \nabla_{\dot\gamma} \nabla_X X, \dot\gamma\rangle = \dot\gamma\big(\langle \nabla_X X, \dot\gamma\rangle\big) - \langle \nabla_X X, \nabla_{\dot\gamma}\dot\gamma\rangle = \dot\gamma\big(\langle \nabla_X X, \dot\gamma\rangle\big)$, by metric compatibility.
- $\|\nabla_{\dot\gamma} X\|^2 - \langle \nabla_{\dot\gamma} X, \dot\gamma\rangle^2 = \|\nabla_{\dot\gamma} X^\perp\|^2$, by Lemma 10.5.

Combining:

$$\frac{\partial}{\partial s}\left(\frac{\langle \nabla_T X, T\rangle}{\|T\|}\right)\bigg|_{s=0} = \dot\gamma\big(\langle \nabla_X X, \dot\gamma\rangle\big) + \|\nabla_{\dot\gamma} X^\perp\|^2 - R(\dot\gamma, X, \dot\gamma, X).$$

Integrating over $t \in [a, b]$: the first term is a total t-derivative and integrates to the boundary term $\langle \nabla_X X, \dot\gamma\rangle\big|_a^b$. This gives the second variation formula. □

Consider the following special case.
Let $\gamma\colon [0,1] \to M$ be a geodesic with $\|\dot\gamma\| = 1$, and let Z be a parallel unit vector field along γ with $Z \perp \dot\gamma$ (so $\nabla_{\dot\gamma} Z = 0$ and $\|Z\| = 1$). Let $f\colon [0,1] \to \mathbb{R}$ be smooth. Define the variation

$$\phi(s,t) = \exp_{\gamma(t)}\big(s \cdot f(t) \cdot Z(t)\big).$$

(Side note: we are using essentially the same variation in Lemma 6.26/exercise 6.3. a)
For each fixed t, the curve $s \mapsto \phi(s,t)$ is a geodesic with initial velocity $f(t)Z(t)$. The variation vector field at $s = 0$ is

$$X(t) = \left.\frac{\partial \phi}{\partial s}\right|_{s=0} = f(t)\, Z(t).$$

Two simplifications:

- Since each s-curve is a geodesic, $\nabla_X X = 0$, so the boundary term in the second variation formula vanishes.
- Since $X = fZ$ with Z parallel, $\nabla_{\dot\gamma} X = \dot f Z$, and since $Z \perp \dot\gamma$, this is also perpendicular to $\dot\gamma$, hence $X^\perp = X$ and
$$\|\nabla_{\dot\gamma} X^\perp\|^2 = \|\dot f Z\|^2 = \dot f^2.$$

For the curvature term, using $\|\dot\gamma\| = \|Z\| = 1$ and $Z \perp \dot\gamma$ (so $\|\dot\gamma \wedge Z\|^2 = 1$):

$$R(\dot\gamma, X, \dot\gamma, X) = f^2\, R(\dot\gamma, Z, \dot\gamma, Z) = f^2\, K(\dot\gamma, Z).$$

Plugging into the second variation formula:

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}s^2}\ell(\phi_s)\right|_{s=0} = \int_0^1 \Big(\dot f^2 - K(\dot\gamma, Z)\, f^2\Big)\, \mathrm{d}t. \tag{10.1}$$

Remark 10.6. Interpretation and connection to Bonnet–Myers
The integrand $\dot f^2 - K f^2$ trades off oscillation in f against positive curvature along γ: rapid oscillation makes $\ell(\phi_s)$ increase to second order, while positive curvature makes it decrease. Concretely, on S^2, a great-circle arc longer than π fails to minimize length: taking $f(t) = \sin(\pi t)$ and Z a parallel field perpendicular to the arc produces $\frac{d^2}{ds^2}\ell|_{s=0} < 0$. The Bonnet–Myers Theorem below turns this into a global theorem, which converts the sectional-curvature input into a Ricci-curvature bound and forces $\operatorname{diam}(M) \leqslant \pi/a$.

Theorem 10.7 (Bonnet–Myers). Let (M^n, g) be a complete, connected Riemannian manifold with Ricci tensor bounded below by some $a > 0$:

$$\operatorname{Ric}(X,X) \geqslant a^2 (n-1)\, \|X\|^2 \qquad \text{for all } X \in \Gamma(TM).$$

Then $\operatorname{diam}(M,g) \leqslant \pi/a$; in particular, M is compact.

Remark 10.8. The bound is sharp: the sphere of radius $1/a$ has $\operatorname{Ric} = a^2(n-1)\, g$ and diameter exactly π/a.

(Lecture 12)

Proof of Theorem 10.7. Suppose for contradiction that $\operatorname{diam}(M,g) > \pi/a$. Pick $p, q \in M$ with $\ell := d(p,q) > \pi/a$. By Hopf–Rinow, Theorem 7.2 there exists a length-minimizing geodesic $\gamma\colon [0,\ell] \to M$ with $\|\dot\gamma\| = 1$, $\gamma(0) = p$, $\gamma(\ell) = q$. We derive a contradiction by constructing endpoint-fixing variations of γ with negative second variation.

Step 1: parallel frame. By Lemma 6.10, there exists a parallel orthonormal frame $\{E_1, \ldots, E_{n-1}, \dot\gamma\}$ along γ, i.e. $\nabla_{\dot\gamma} E_i = 0$ and $\{E_i\}_{i=1}^{n-1}$ is orthonormal and perpendicular to $\dot\gamma$ at each point.
Step 2: test variations. For each $i = 1, \ldots, n-1$, define

$$\phi^{(i)}(s,t) := \exp_{\gamma(t)}\big(s \cdot f(t) \cdot E_i(t)\big), \qquad f(t) := \sin\!\big(\tfrac{\pi t}{\ell}\big).$$

Since $f(0) = f(\ell) = 0$, each variation fixes the endpoints. By the test-variation formula (10.1) derived earlier,

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\,\ell(\phi_s^{(i)})\Big|_{s=0} = \int_0^\ell \big(\dot f^2 - K(\dot\gamma, E_i)\, f^2\big)\,\mathrm{d}t.$$

Step 3: integration by parts. Since $\ddot f = -(\pi/\ell)^2 f$ and f vanishes at the endpoints, integration by parts gives

$$\int_0^\ell \dot f^2\,\mathrm{d}t = -\int_0^\ell f\ddot f\,\mathrm{d}t = \frac{\pi^2}{\ell^2}\int_0^\ell f^2\,\mathrm{d}t.$$

Hence

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\,\ell(\phi_s^{(i)})\Big|_{s=0} = \int_0^\ell \Big(\tfrac{\pi^2}{\ell^2} - K(\dot\gamma, E_i)\Big)\, f^2\,\mathrm{d}t.$$

Step 4: summing over i. Summing over $i = 1, \ldots, n-1$ and using $\sum_{i=1}^{n-1} K(\dot\gamma, E_i) = \mathrm{Ric}(\dot\gamma,\dot\gamma) \geqslant a^2(n-1)$:

$$\begin{aligned}\sum_{i=1}^{n-1}\frac{\mathrm{d}^2}{\mathrm{d}s^2}\,\ell(\phi_s^{(i)})\Big|_{s=0} &= \int_0^\ell \Big(\tfrac{\pi^2(n-1)}{\ell^2} - \mathrm{Ric}(\dot\gamma,\dot\gamma)\Big)\, f^2\,\mathrm{d}t\\ &\leqslant (n-1)\int_0^\ell \Big(\tfrac{\pi^2}{\ell^2} - a^2\Big)\, f^2\,\mathrm{d}t\\ &< 0,\end{aligned}$$

since $\ell > \pi/a$ gives $\pi^2/\ell^2 < a^2$. Hence at least one i has $\frac{\mathrm{d}^2}{\mathrm{d}s^2}\ell(\phi_s^{(i)})|_{s=0} < 0$, contradicting the assumption that γ is length-minimizing. □

10.2 Einstein manifolds and homogeneity

The Bonnet–Myers bound is attained by the sphere, where $\mathrm{Ric} = a^2(n-1)\,g$ is a constant multiple of the metric. This proportionality is the defining feature of an important class of manifolds.

Definition 10.9 (Einstein tensor and Einstein manifolds). The *Einstein tensor* of (M, g) is the $(0,2)$-tensor

$$G := \mathrm{Ric} - \tfrac{1}{2}\,S\,g,$$

where S is the scalar curvature. A Riemannian manifold (M, g) is called an *Einstein manifold* if there exists a smooth function $\Lambda\colon M \to \mathbb{R}$ with $\mathrm{Ric} = \Lambda\, g$.

Lemma 10.10 (Properties of the Einstein tensor and Einstein manifolds).

(a) The Einstein tensor is divergence-free: in local coordinates,

$$g^{ab}\,\nabla_a G_{bc} = 0.$$

(b) If (M, g) is an Einstein manifold of dimension $\dim M \geqslant 3$, then Λ is constant on each connected component of M.

(c) If (M, g) is a connected Einstein manifold with $\dim M = 3$, then (M, g) has constant sectional curvature.

Proof. See Exercise 11.2. □

Remark 10.11. In general relativity, a vacuum region of spacetime is modelled by a Lorentzian manifold satisfying $G = \Lambda g$, where Λ is the *cosmological constant*. By parts (b) and (c) of the lemma, non-trivial vacuum spacetimes exist only in dimension $\geqslant 4$.

Highly symmetric manifolds are automatically Einstein. To state this precisely, we first define a notion of symmetry.

Definition 10.12 (k-homogeneous Riemannian manifold). Let (M, g) be a Riemannian manifold and $k \geqslant 1$ an integer. (M, g) is called *k-homogeneous* if for any two k-tuples $(p_1, \dots, p_k)$ and $(q_1, \dots, q_k)$ in M^k satisfying $\mathrm{dist}_g(p_i, p_j) = \mathrm{dist}_g(q_i, q_j)$ for all $i, j \in \{1, \dots, k\}$, there exists an isometry $F\colon M \to M$ with $F(p_i) = q_i$ for all i. The case $k = 1$ recovers the usual notion of a homogeneous Riemannian manifold (Exercise 8.2).

Lemma 10.13 (Properties of k-homogeneous manifolds). Let (M, g) be a Riemannian manifold.

(a) If (M, g) is k-homogeneous for some $k \geqslant 2$, then it is also $(k-1)$-homogeneous. In particular, every k-homogeneous space is complete.

(b) If (M, g) is 2-homogeneous, it is isotropic (Exercise 8.3). If moreover $\dim M \geqslant 3$, then (M, g) is Einstein (Definition 10.9): $\mathrm{Ric} = \lambda g$ for some $\lambda \in \mathbb{R}$.

(c) If (M, g) is 3-homogeneous, then it has constant sectional curvature.

Proof. See Exercise 12.2. □

10.3 Jacobi fields and conjugate points

Let (M, g) be a complete Riemannian manifold and $p \in M$. We want to understand when $\exp_p$ stops being a diffeomorphism. There are two possible failure modes:

- *Global failure (not injective):* e.g. on the cylinder, two distinct vectors in T_pM can have the same exponential image.
- *Local failure (differential degenerates):* the map $\mathrm{d}\exp_p\big|_v \colon T_v(T_pM) \to T_{\exp_p(v)}M$ fails to be invertible for some $v \in T_pM$.

In this section we focus on the local failure. By the Gauss lemma (Lemma 6.35), $\mathrm{d}\exp_p\big|_v$ is the identity on the radial direction, so $\mathrm{d}\exp_p\big|_v(w) = 0$ forces $w \perp v$. Geometrically, the kernel of $\mathrm{d}\exp_p\big|_v$ measures, infinitesimally, families of geodesics starting at p that reintersect at $\exp_p(v)$.
To study this, fix $v \in T_pM$ and let $\gamma(t) = \exp_p(vt)$. For a smooth curve $w\colon (-\epsilon, \epsilon) \to T_pM$ (i.e. a curve in the tangent space, parametrizing a family of tangent vectors at p) with $w(0) = v$, the map

$$\phi(s, t) = \exp_p(w(s) \cdot t)$$

defines a variation of γ in which each s-curve $t \mapsto \phi(s, t)$ is a geodesic from p with initial velocity $w(s)$. (A very similar variation appears in Lemma 6.26, Exercise 6.3(a).) Taking the linear family $w(s) = v + s\,w_0$ for $w_0 \in T_pM$, the variation vector field at $t = 1$ is

$$\left.\frac{\partial \phi}{\partial s}\right|_{(s,t)=(0,1)} = \mathrm{d}\exp_p\big|_v(w_0).$$

Hence understanding when $\mathrm{d}\exp_p\big|_v$ has nontrivial kernel reduces to understanding when variation vector fields of geodesic variations vanish at $t = 1$.

Since each s-curve of ϕ is a geodesic, the geodesic deviation equation (Lemma 8.11) applies and the variation vector field $X = \partial\phi/\partial s$ satisfies

$$\nabla_T \nabla_T X - R(T, X)\,T = 0, \qquad T = \partial\phi/\partial t.$$

This motivates the following definition.

Definition 10.14 (Jacobi vector field). Let $\gamma\colon [0, \ell] \to (M, g)$ be a geodesic. A vector field J along γ is called a *Jacobi vector field* if it satisfies the *Jacobi equation*

$$\nabla_{\dot\gamma} \nabla_{\dot\gamma} J - R(\dot\gamma, J)\,\dot\gamma = 0. \tag{10.2}$$

This is a second-order linear ODE; a solution is uniquely determined by the initial data $J(0)$ and $\nabla_{\dot\gamma} J(0)$.

In dimension 2, writing the metric in geodesic polar coordinates as $g = \mathrm{d}r^2 + f(r,\theta)^2\,\mathrm{d}\theta^2$, the Jacobi equation reduces to the scalar ODE $\partial_r^2 f + K f = 0$, where K is the Gauss curvature (Exercise 10.1).

Remark 10.15. By the geodesic deviation equation (Lemma 8.11), the variation vector field $X = \partial\phi/\partial s$ of any variation by geodesics is a Jacobi field. We will see later (Lemma 10.20) that the converse also holds: every Jacobi field arises as the variation vector field of some variation by geodesics.

Example 10.16 (Trivial Jacobi fields).

- Along any geodesic γ, the field $J(t) = (At + B)\,\dot\gamma(t)$ is a Jacobi field for any $A, B \in \mathbb{R}$. Indeed, $\nabla_{\dot\gamma} J = A\,\dot\gamma$, hence $\nabla_{\dot\gamma}\nabla_{\dot\gamma} J = 0$, and $R(\dot\gamma,\dot\gamma)\dot\gamma = 0$ by antisymmetry, so (10.2) holds. In particular, $\dot\gamma$ and $t\,\dot\gamma$ are Jacobi fields.
- On $(\mathbb{R}^n, g_E)$, the curvature vanishes ($R \equiv 0$) and the Levi-Civita connection reduces to the standard directional derivative ($\Gamma^k_{ij} = 0$), so the Jacobi equation becomes

  $$0 = \nabla_{\dot\gamma}\nabla_{\dot\gamma} J - R(\dot\gamma, J)\,\dot\gamma = \ddot J.$$

  Integrating twice, all Jacobi fields are of the form $J(t) = t\,V_1 + V_0$ with $V_0, V_1 \in \mathbb{R}^n$ constant.

We introduce the notation $\dot J := \nabla_{\dot\gamma} J$.

Lemma 10.17 (Tangential component of a Jacobi field is affine). Let J be a Jacobi field along a geodesic γ. Then

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle J, \dot\gamma\rangle = 0,$$

so $\langle J, \dot\gamma\rangle(t) = At + B$ is an affine function of t.

Proof. Since γ is a geodesic, $\nabla_{\dot\gamma}\dot\gamma = 0$. By metric compatibility,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle J, \dot\gamma\rangle = \langle \nabla_{\dot\gamma} J, \dot\gamma\rangle + \langle J, \nabla_{\dot\gamma}\dot\gamma\rangle = \langle \dot J, \dot\gamma\rangle.$$

Differentiating once more and using the Jacobi equation (10.2):

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle J, \dot\gamma\rangle = \langle \nabla_{\dot\gamma}\dot J, \dot\gamma\rangle = \langle R(\dot\gamma, J)\,\dot\gamma, \dot\gamma\rangle = 0,$$

where the last equality holds by the antisymmetry of R in its last two arguments. □

Corollary 10.18 (Perpendicular Jacobi fields stay perpendicular). Let J be a Jacobi field along a geodesic γ. If $J(0) \perp \dot\gamma(0)$ and $\dot J(0) \perp \dot\gamma(0)$, then $J(t) \perp \dot\gamma(t)$ for all t.

Proof. By Lemma 10.17, $f(t) := \langle J(t), \dot\gamma(t)\rangle = At + B$ for some $A, B \in \mathbb{R}$. The hypotheses give

$$B = f(0) = \langle J(0), \dot\gamma(0)\rangle = 0, \qquad A = f'(0) = \langle \dot J(0), \dot\gamma(0)\rangle = 0,$$

where we used $\frac{d}{dt}\langle J, \dot\gamma\rangle = \langle \dot J, \dot\gamma\rangle$ from the proof of Lemma 10.17. Hence $f \equiv 0$, i.e. $J(t) \perp \dot\gamma(t)$ for all t. □

Remark 10.19 (Dimension count of Jacobi fields). The space of Jacobi fields along γ is $2n$-dimensional, parametrized by the initial data $(J(0), \dot J(0)) \in T_{\gamma(0)}M \times T_{\gamma(0)}M$. By Corollary 10.18 and Lemma 10.17, this splits as

$$\underbrace{2}_{\text{tangential: } (At+B)\,\dot\gamma} \quad + \quad \underbrace{2(n-1)}_{\text{perpendicular: } J\perp\dot\gamma}.$$

Restricting further to Jacobi fields with $J(0) = 0$ removes n degrees of freedom (the choice of $J(0)$), leaving an n-dimensional space; the perpendicular ones among these form an $(n-1)$-dimensional subspace (since the unique tangential Jacobi field with $J(0) = 0$ is $t \cdot \dot\gamma$, up to scaling).

Hence the interesting Jacobi fields are those perpendicular to $\dot\gamma$: the tangential part is a trivial $(At + B)\dot\gamma$ that carries no curvature information, so we restrict our attention to Jacobi fields $J \perp \dot\gamma$ from now on.

Lemma 10.20 (Jacobi fields come from variations by geodesics (restricted version)). Let $\gamma : [0, \ell] \to M$ be a geodesic and let J be a Jacobi field along γ with $J(0) = 0$. Then there exists a variation by geodesics

$$\phi : (-\epsilon, \epsilon) \times [0, \ell] \to M, \qquad \phi(s, 0) = \gamma(0) \text{ for all } s,$$

whose variation vector field along γ recovers J:

$$\left.\frac{\partial \phi}{\partial s}\right|_{s=0} = J.$$

Proof. Let $p = \gamma(0)$, $v = \dot\gamma(0)$, and define $W : (-\epsilon, \epsilon) \to T_pM$ by

$$W(s) := v + s \cdot \dot J(0),$$

so $W(0) = v$ and $W'(0) = \dot J(0)$. Set

$$\phi(s, t) := \exp_p(W(s) \cdot t).$$

Then $\phi(s, 0) = p$ for all s, and for each fixed s the curve $t \mapsto \phi(s, t)$ is a geodesic from p with initial velocity $W(s)$. So ϕ is a variation by geodesics, and

$$\widetilde{J}(t) := \frac{\partial \phi}{\partial s}(0, t) = \mathrm{d}\exp_p|_{vt}(W'(0) \cdot t)$$

is a Jacobi field along γ by the geodesic deviation equation (Lemma 8.11).
By uniqueness of solutions to the Jacobi equation, it suffices to verify that $\widetilde{J}$ and J have the same initial data:

- $\widetilde{J}(0) = \mathrm{d}\exp_p|_0(0) = 0 = J(0)$.
- $\dot{\widetilde{J}}(0) = W'(0) = \dot J(0)$, since $\mathrm{d}\exp_p|_0 = \mathrm{id}$ (Lemma 6.23).

Hence $\widetilde{J} = J$ on $[0, \ell]$. □

Remark 10.21. The hypothesis $J(0) = 0$ is essential to this construction, since the variation $\phi(s,t) = \exp_p(W(s) \cdot t)$ has all geodesics starting at the same point p, forcing $\widetilde{J}(0) = 0$. The general case ($J(0) \neq 0$) also holds — every Jacobi field is the variation vector field of some variation by geodesics — but the construction is more involved and requires varying the starting point as well.

By Lemma 6.10, there exists an orthonormal frame $\{E_1, \dots, E_n\}$ along γ that is parallel ($\nabla_{\dot\gamma} E_j = 0$) with E_1 collinear with $\dot\gamma$. Relabel the perpendicular part as $e_i := E_{i+1}$ for $i = 1, \dots, n-1$, so $\{e_i\}_{i=1}^{n-1}$ is a parallel orthonormal frame along γ with $e_i \perp \dot\gamma$.
For a Jacobi field $J = J^i e_i$ perpendicular to $\dot\gamma$, since the frame is parallel:

$$\nabla_{\dot\gamma} J = \dot J^i\, e_i, \qquad \nabla_{\dot\gamma}\nabla_{\dot\gamma} J = \ddot J^i\, e_i.$$

Expanding the Jacobi equation in this frame:

$$0 = \nabla_{\dot\gamma}\nabla_{\dot\gamma} J - R(\dot\gamma, J)\,\dot\gamma = \ddot J^i\, e_i - J^i\, R(\dot\gamma, e_i)\,\dot\gamma = \ddot J^k\, e_k - \langle R(\dot\gamma, e_i)\,\dot\gamma,\, e_k\rangle\, J^i\, e_k,$$

where in the last step we expanded each vector $R(\dot\gamma, e_i)\,\dot\gamma$ in the orthonormal basis $\{e_k\}$. Reading off the e_k-component yields the system of ODEs

$$\ddot J^k + a_i^k\, J^i = 0, \qquad a_i^k := -\langle R(\dot\gamma, e_i)\,\dot\gamma,\, e_k\rangle = R(\dot\gamma, e_i, \dot\gamma, e_k). \tag{10.3}$$

The matrix (a_i^k) is symmetric (by the pairwise symmetry of R) and depends only on the curvature along γ.

Definition 10.22 (Conjugate point). Let $\gamma\colon [a,b] \to M$ be a geodesic and $t_0, t_1 \in [a,b]$ with $t_0 \neq t_1$. The point $\gamma(t_1)$ is said to be *conjugate* to $\gamma(t_0)$ along γ if there exists a nonzero Jacobi field J along γ with $J(t_0) = 0$ and $J(t_1) = 0$. The *multiplicity* of $\gamma(t_1)$ as a conjugate point is the number of linearly independent such Jacobi fields.

Remark 10.23. The multiplicity is at most $n-1$: any Jacobi field with $J(t_0) = J(t_1) = 0$ must be perpendicular to $\dot\gamma$ (since the tangential part $(At+B)\dot\gamma$ can vanish at most once unless identically zero), and the space of perpendicular Jacobi fields with $J(t_0) = 0$ is $(n-1)$-dimensional.

Example 10.24 (Conjugate points on the round sphere). On the round sphere S^n, the conjugate point of p along any geodesic from p is the antipodal point $-p$, with multiplicity $n-1$.

Remark 10.25. If $\gamma\colon [0,\ell] \to M$ is a geodesic with $\dot\gamma(0) = v$, then for $\mathrm{d}\exp_p\big|_{tv}$:

$$\dim\ker = \text{multiplicity of } \gamma(t) \text{ as a conjugate point of } \gamma(0).$$

(Lecture 13)

Theorem 10.26 (Conjugate points obstruct length minimization). Let $\gamma\colon [0,\ell] \to (M,g)$ be a geodesic. If there exist $t_0, t_1 \in (0,\ell)$ with $t_0 < t_1$ such that $\gamma(t_0)$ and $\gamma(t_1)$ are conjugate along γ, then γ is not length-minimizing between $\gamma(0)$ and $\gamma(\ell)$.

Example 10.27. On the round sphere S^n, any great-circle arc longer than π (i.e. extending past an antipodal pair) is not length-minimizing.

Proof. Without loss of generality $\|\dot\gamma\| = 1$. Let J be a nonzero Jacobi field with $J(t_0) = J(t_1) = 0$. We construct an endpoint-fixing variation ϕ of γ such that

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\,\ell(\phi_s)\Big|_{s=0} < 0;$$

the first variation vanishes automatically since γ is a geodesic.
Step 1: a first attempt with Y from J. Define the continuous, piecewise-smooth vector field along γ

$$Y(t) := \begin{cases} J(t), & t \in [t_0, t_1], \\ 0, & t \in [0, t_0] \cup [t_1, \ell], \end{cases}$$

and let $\phi(s,t) := \exp_{\gamma(t)}\big(s\,Y(t)\big)$. This is smooth in s, piecewise smooth in t, and the variation formulas extend to such variations. The variation vector field at $s = 0$ is $X|_{s=0} = Y$, and since each s-curve $s \mapsto \phi(s,t)$ is a geodesic, $\nabla_X X = 0$ at $s = 0$. By the 2nd variation formula:

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}s^2}\,\ell(\phi_s)\Big|_{s=0} &= \int_0^{\ell} \big(\|\nabla_{\dot\gamma} Y\|^2 - R(\dot\gamma, Y, \dot\gamma, Y)\big)\,\mathrm{d}t \\ &= \int_{t_0}^{t_1} \big(\|\nabla_{\dot\gamma} J\|^2 - R(\dot\gamma, J, \dot\gamma, J)\big)\,\mathrm{d}t. \end{aligned}$$

Step 2: this first attempt gives zero. Using the Jacobi equation $\nabla_{\dot\gamma}\nabla_{\dot\gamma} J = R(\dot\gamma, J)\dot\gamma$ and integration by parts (noting $J(t_0) = J(t_1) = 0$):

$$\begin{aligned} \int_{t_0}^{t_1} \|\nabla_{\dot\gamma} J\|^2\,\mathrm{d}t &= \int_{t_0}^{t_1} \Big(\tfrac{\mathrm{d}}{\mathrm{d}t}\langle \nabla_{\dot\gamma} J, J\rangle - \langle \nabla_{\dot\gamma}\nabla_{\dot\gamma} J, J\rangle\Big)\,\mathrm{d}t \\ &= 0 - \int_{t_0}^{t_1} \langle R(\dot\gamma, J)\dot\gamma, J\rangle\,\mathrm{d}t = \int_{t_0}^{t_1} R(\dot\gamma, J, \dot\gamma, J)\,\mathrm{d}t. \end{aligned}$$

Hence $\frac{\mathrm{d}^2}{\mathrm{d}s^2}\,\ell(\phi_s)\big|_{s=0} = 0$.
Step 3: smoothing the corner at t_1 makes the second variation strictly negative. The issue is the corner of Y at $t = t_1$ (and similarly at t_0). Note $\dot J(t_1) \neq 0$ (recall $\dot J := \nabla_{\dot\gamma} J$), otherwise $J \equiv 0$ by uniqueness. For small $\delta \ll \varepsilon \ll 1$, replace Y on $[t_1 - \delta, t_1 + \varepsilon]$ by a linear interpolation:

$$\tilde Y(t) := \begin{cases} Y(t), & t \notin [t_1 - \delta,\ t_1 + \varepsilon], \\ Y(t_1 - \delta) \cdot \dfrac{(t_1 + \varepsilon) - t}{\varepsilon + \delta}, & t \in [t_1 - \delta,\ t_1 + \varepsilon]. \end{cases}$$

Taylor expansion gives $\|Y(t_1 - \delta)\| = \|\dot J(t_1)\| \cdot \delta + O(\delta^2)$, so $\|Y - \tilde Y\|(t) \lesssim \delta$ uniformly. Set $\tilde\phi(s,t) = \exp_{\gamma(t)}(s\,\tilde Y(t))$, which agrees with ϕ outside $[t_1 - \delta, t_1 + \varepsilon]$.
Step 4: comparing the second variations. The difference $\frac{\mathrm{d}^2}{\mathrm{d}s^2}\ell(\tilde\phi_s)|_{s=0} - \frac{\mathrm{d}^2}{\mathrm{d}s^2}\ell(\phi_s)|_{s=0}$ is an integral over $[t_1 - \delta, t_1 + \varepsilon]$. The curvature term contributes at most $O(\delta)$:

$$\left|\int_{t_1 - \delta}^{t_1 + \varepsilon} \big(R(\dot\gamma, Y, \dot\gamma, Y) - R(\dot\gamma, \tilde Y, \dot\gamma, \tilde Y)\big)\,\mathrm{d}t\right| \lesssim \delta,$$

since the integrand is bounded by $|K| \cdot \|Y - \tilde Y\| \cdot (\|Y\| + \|\tilde Y\|) \lesssim \delta$. The kinetic terms give the dominant contribution: on $[t_1 - \delta, t_1]$,

$$\|\nabla_{\dot\gamma} Y\| = \|\dot J(t_1)\| + O(\delta), \qquad \|\nabla_{\dot\gamma} \tilde Y\| = \frac{\|\dot J(t_1)\|}{\varepsilon + \delta}\,\delta + O\big(\tfrac{\delta}{\varepsilon}\big),$$

while on $[t_1, t_1+\varepsilon]$, $\nabla_{\dot\gamma} Y = 0$ and $\|\nabla_{\dot\gamma}\tilde Y\| = \frac{\|\dot J(t_1)\|\,\delta}{\varepsilon+\delta} + O(\frac{\delta^2}{\varepsilon})$. Combining,

$$\int_{t_1-\delta}^{t_1+\varepsilon} \left(\|\nabla_{\dot\gamma} Y\|^2 - \|\nabla_{\dot\gamma}\tilde Y\|^2\right) \mathrm{d}t = -\|\dot J(t_1)\|^2 + O(\varepsilon\delta).$$

Hence, for $\delta \ll \varepsilon \ll 1$:

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\,\ell(\tilde\phi_s)\Big|_{s=0} = 0 - \|\dot J(t_1)\|^2 + O(\varepsilon\delta) + O(\delta) < 0. \qquad \square$$

We have the following comparison principle: *larger curvature produces smaller Jacobi fields*. Indeed, the scalar equation $f'' + K f = 0$ in the theorem below is the Jacobi equation (10.2) in dimension 2, written for the amplitude in a parallel normal frame.

Theorem 10.28 (Sturm's comparison theorem). Let $f, \tilde f\colon [0,\ell] \to \mathbb{R}$ satisfy $f'' + K f = 0$ and $\tilde f'' + \widetilde K \tilde f = 0$ with $K, \widetilde K \in C^0([0,\ell])$, and assume

$$f(0) = \tilde f(0) = 0, \qquad f'(0) = \tilde f'(0) > 0, \qquad \tilde f \neq 0 \text{ on } (0,\ell].$$

If $\widetilde K(t) \geqslant K(t)$ for all $t \in [0,\ell]$, then

$$\tilde f(t) \leqslant f(t) \quad \text{on } [0,\ell].$$

The higher-dimensional generalization of Theorem 10.28, replacing the scalar ODE by the full Jacobi equation, is:

Theorem 10.29 (Rauch comparison theorem). Let $\gamma\colon [0,a] \to M^n$ and $\tilde\gamma\colon [0,a] \to \widetilde M^{n'}$ $(n' \geqslant n)$ be unit-speed geodesics, and let $J, \tilde J$ be Jacobi fields (Definition 10.14) along them with

$$J(0) = \tilde J(0) = 0, \qquad \langle \dot J(0), \dot\gamma(0)\rangle = \langle \dot{\tilde J}(0), \dot{\tilde\gamma}(0)\rangle, \qquad \|\dot J(0)\| = \|\dot{\tilde J}(0)\|.$$

Assume $\tilde\gamma$ has no conjugate points (Definition 10.22) on $(0,a]$, and that for all $V \in T_{\gamma(t)}M$, $\widetilde V \in T_{\tilde\gamma(t)}\widetilde M$,

$$\widetilde K(\widetilde V, \dot{\tilde\gamma}(t)) \geqslant K(V, \dot\gamma(t)).$$

Then $\|\tilde J(t)\| \leqslant \|J(t)\|$ for all $t \in [0,a]$. Moreover, if $\|\tilde J(a)\| = \|J(a)\|$, then $K(J(t),\dot\gamma(t)) = \widetilde K(\tilde J(t), \dot{\tilde\gamma}(t))$ for all $t \in [0,a]$.

Rauch (Theorem 10.29) is an *infinitesimal* comparison statement (Jacobi fields encode first-order deviation of geodesics). The corresponding statement for finite geodesic triangles is:

Theorem 10.30 (Toponogov's theorem). Let (M,g) have sectional curvature $K \geqslant \rho$ for some $\rho \in \mathbb{R}$. Let $\triangle_{pqr}$ be a geodesic triangle in M with pq a minimal geodesic (and, if $\rho > 0$, $\ell(pr) < \pi/\sqrt{\rho}$). Let $\triangle_{p'q'r'}$ be the comparison triangle in the model 2D space of constant curvature ρ, defined by

$$\ell(p'q') = \ell(pq), \quad \ell(p'r') = \ell(pr), \quad \angle_{q'p'r'} = \angle_{qpr}.$$

Then $\ell(qr) \leqslant \ell(q'r')$.

10.4 The Cartan–Hadamard theorem

We now study the topology of manifolds with non-positive sectional curvature. The key analytic input is that Jacobi field norms grow convexly.

Theorem 10.31 (Convexity of Jacobi field norm)**.** Let (M, g) have sectional curvature $K \leqslant 0$ and let γ be a geodesic. Then for any Jacobi field J along γ, the function $t \mapsto \|J(t)\|$ is convex.

Proof. First, $\|J\|^2$ has convex second derivative:

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\|J\|^2 &= 2\,\langle \nabla_{\dot\gamma}\nabla_{\dot\gamma}J,\ J\rangle + 2\,\|\nabla_{\dot\gamma}J\|^2\\ &= 2\,\underbrace{\langle R(\dot\gamma, J)\,\dot\gamma,\ J\rangle}_{\geqslant 0\ (\text{since } K\leqslant 0)} + 2\,\|\nabla_{\dot\gamma}J\|^2 \ \geqslant\ 2\,\|\nabla_{\dot\gamma}J\|^2,\end{aligned}$$

where we used the Jacobi equation and the sign of the curvature term: $\langle R(\dot\gamma, J)\dot\gamma, J\rangle = -K(\dot\gamma, J)\,\|\dot\gamma \wedge J\|^2 \geqslant 0$.

For $\|J\|$ itself, the chain rule gives

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\|J\| = \frac{\frac{\mathrm{d}^2}{\mathrm{d}t^2}\|J\|^2}{2\,\|J\|} - \frac{\left(\frac{\mathrm{d}}{\mathrm{d}t}\|J\|^2\right)^2}{4\,\|J\|^3} \geqslant \frac{\|\nabla_{\dot\gamma}J\|^2}{\|J\|} - \frac{\langle \nabla_{\dot\gamma}J, J\rangle^2}{\|J\|^3} \geqslant 0,$$

where the last inequality is Cauchy–Schwarz: $\langle \nabla_{\dot\gamma}J, J\rangle^2 \leqslant \|\nabla_{\dot\gamma}J\|^2 \cdot \|J\|^2$. □

Theorem 10.32 (Exponential map is length non-decreasing in non-positive curvature)**.** Let (M, g) be complete and connected with $K \leqslant 0$. Then for all $p \in M$ and all $v, w \in T_pM$,

$$\|\mathrm{d}\exp_p|_v(w)\| \geqslant \|w\|.$$

In particular, $\mathrm{d}\exp_p|_v$ is injective for every v, so $\exp_p\colon T_pM \to M$ is a local diffeomorphism.

Proof. For the variation $\phi(t, s) = \exp_p(t(v + sw))$, the variation vector field $J(t) := \frac{\partial\phi}{\partial s}\big|_{s=0}$ is a Jacobi field along the geodesic $t \mapsto \exp_p(tv)$, and

$$J(t) = \mathrm{d}\exp_p|_{tv}(tw),$$

with $J(0) = 0$ and $\dot J(0) = w$ (the latter computed in normal coordinates around p, using $\mathrm{d}\exp_p|_0 = \mathrm{id}$). By Theorem 10.31, $t \mapsto \|J(t)\|$ is convex. A convex function f with $f(0) = 0$ satisfies $f(t) \geqslant t \cdot f'(0^+)$ for $t \geqslant 0$, so

$$\|J(t)\| \geqslant t \cdot \|\dot J(0)\| = t \cdot \|w\|.$$

Evaluating at $t = 1$: $\|\mathrm{d}\exp_p|_v(w)\| = \|J(1)\| \geqslant \|w\|$. □

Definition 10.33 (Smooth covering map)**.** A smooth map $\pi\colon N \to M$ between smooth manifolds is a *smooth covering map* if every point $q \in M$ has an open neighborhood U whose preimage $\pi^{-1}(U)$ is a disjoint union of open sets $\{\widetilde U_i\}_{i\in I}$ in N, each mapped diffeomorphically onto U by π:

$$\pi^{-1}(U) = \bigsqcup_{i\in I} \widetilde U_i.$$

Theorem 10.34 (Cartan–Hadamard)**.** If (M, g) is connected, complete, and has $K \leqslant 0$, then $\exp_p\colon T_pM \to M$ is a smooth covering map for every $p \in M$.

Remark 10.35. Topological consequences

- If M is simply connected, then $\exp_p$ is a diffeomorphism, so $M \cong \mathbb{R}^n$.
- In particular, no compact, simply connected manifold of dimension $\geqslant 1$ admits a metric with non-positive sectional curvature.

10.5 The divergence and the divergence theorem

Recall the volume form dV_g and the integral $\int_M h\, dV_g$ from Section 3.2. We can now define the divergence operator and state the Riemannian divergence theorem.

Definition 10.36 (Divergence). The *divergence* of a vector field $X \in \Gamma(TM)$ is $\operatorname{div} X := \operatorname{tr}(\nabla X) = (\nabla_i X)^i$. For a 1-form ω, set $\operatorname{div}\omega := \operatorname{div}(\omega^\sharp) = g^{ij}(\nabla_i \omega)_j$.

See also Example 4.24 and Lemma 10.10(a).

Lemma 10.37 (Divergence theorem). Let (M, g) be a Riemannian manifold, ω a smooth 1-form on M, and $\Omega \subset M$ an open domain with compact closure and piecewise smooth boundary $\partial\Omega$. Let $\hat{n}$ denote the outward unit normal and $\bar{g}$ the induced metric on $\partial\Omega$. Then

$$\int_\Omega \operatorname{div}\omega \; dV_g = \int_{\partial\Omega} \omega(\hat{n}) \; dV_{\bar{g}}. \tag{10.4}$$

Moreover, in any local coordinate system,

$$\operatorname{div}\omega = \frac{1}{\sqrt{\det g}}\, \partial_i\big(g^{ij}\sqrt{\det g}\,\omega_j\big).$$

Proof. See Exercise 13.3. □

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